STOCHASTIC DOMINATIONS FOR FK PERCOLATION AND SHARP THINNING THRESHOLDS FOR THE ISING ENERGY FIELD

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ABSTRACT. At first glance, one would imagine that the energy field of the Ising model, the set of edges whose endpoints share the same spin, is stochastically monotone as a function of the coupling constants. However, this is not generally the case. In this paper, we introduce two weaker notions of stochastic domination that make this result true: p–weak and p–weak† domination. Both of these notions depend on a parameter p and we find the optimal values p and $p^\dagger$ so that these dominations hold.

One of the key ingredient to obtain some of the results is a new stochastic domination relating FK percolations with different parameters $q, \tilde{q} \geq 1$ that is of independent interest.

1. INTRODUCTION

The Ising model has become a fundamental model in probability theory. For a given finite graph $G = (V, E)$ and coupling constants $J \in (\mathbb{R}^+)^E$, the *(free) Ising model on G* is the probability distribution $\mathbb{P}_J$ on $\{\pm 1\}^V$ satisfying

$$\mathbb{P}_J(\sigma) \propto e^{-H_J(\xi(\sigma))}$$

for any spin configuration $\sigma : V \to \{\pm 1\}$, where the *Ising energy field* $\xi(\sigma)$ is the percolation configuration defined at any given edge $(xy) \in E$ as $(\xi(\sigma))_{xy} = \mathbf{1}_{\sigma(x)=\sigma(y)}$, whose energy is defined by

$$H_J(\xi) = -2 \sum_{e \in E} J_e \xi_e.$$

In this paper, we will only deal with free boundary conditions.

Since $H_J(\xi)$ is minimized when ξ is everywhere equal to 1 (the fully aligned state), one would naturally expect that the law of $\xi(\sigma)$ stochastically increases in J. This expectation is correct at the level of one edge-marginals by the monotonicity of the spin correlations, but is false for the joint law. There are finite graphs for which increasing the coupling constants makes some increasing event in the energy field less likely, a phenomenon that persists even when J is constant and increases across all edges [Häg96]. The energy field also need not satisfy the FKG property (also known as positive association). Thus the failure is not a defect of the usual proof methods: it is a genuine obstruction to treating the full energy field as a monotone percolation model. In fact, as recently summarized by Klausen [RK22], many natural observables of the Ising model fail to exhibit expected monotonicity and positive association. This issue is usually circumvented by the use of the monotonicity of the spin correlations, the FKG property of the spins, or by restricting the set of observables of the energy field as developed in [Cam93], which gives a generalized criteria for the observables that are indeed increasing in J.

The point of view of this paper is to measure this failure of monotonicity rather than to bypass it. Given two laws μ and ν on edge configurations, suppose that the expected domination $\mu \preccurlyeq \nu$ is false. We ask whether it becomes true after applying the same independent thinning to both configurations: is there a *strictly positive* parameter p such that

$$\mu \cap \mathrm{Ber}_p \preccurlyeq \nu \cap \mathrm{Ber}_p,$$

a property that we call *weak domination*, and what is the largest p for which this holds? We also introduce an intermediate notion, weak† domination, which requires weak domination for the set of open edges, and weak domination in the reverse direction for the set of closed edges: it is therefore a property which is symmetric upon taking the dual measure. Lying between classical stochastic domination and weak domination, it raises as well the question of the optimal parameter p at which this weak† domination holds. In both cases, the parameter

p is therefore not merely a technical loss: it measures how much monotonicity remains in a field whose full law is not monotone.

For the Ising energy field these questions have two exact answers. The first threshold is $p(J) = 1 - e^{-2J}$. At this value, the Edwards–Sokal coupling identifies $\mathcal{E}_J \cap \mathrm{Ber}_{p(J)}$ with the FK–Ising random-cluster model, which explains why this threshold is natural rather than accidental. It is the largest thinning parameter for which weak monotonicity and weak FKG hold uniformly over finite graphs. The second threshold is $p^\dagger(J) = 1 - e^{-4J}$. It appears when one asks for a version of the same statement that is also stable under complements, namely weak† domination and weak† FKG. The corresponding field $\mathcal{E}_J^\dagger$ has an abstract dual high-temperature expansion, see (4.23), and is closely related to the dual percolation structures arising in the double-random-current couplings of [ALHL26]. From the exact relations that it satisfies, see for example (4.29), and the fact that it arises as a natural threshold, it seems natural to expect other interesting connections to the many models related to the Ising model, see [HJK25] for a framework where these models are connected.

Parallel to our Ising results, we establish some (surprisingly unknown) stochastic dominations concerning FK percolation, a model prominent in Probability theory also known as the random-cluster model. While FK percolation is a standard tool for studying Potts and Ising models on $\mathbb{Z}^d$ and other lattices (see [Gri06] for a comprehensive treatment), many of its core algebraic properties hold for arbitrary finite graphs and it is often studied over general graphs, see for example [BGJ96] a detailed analysis of the FK model on the complete graph. To the best of our knowledge, our main FK result is the first to provide a stochastic inequality that "mixes" different q parameters in an exact way, except for the well-known monotonicity in the q–parameter. Note that Grimmett obtained strict improvements of this monotonicity on lattices, which allowed him to prove strict monotonicity in q of the critical point $p_c(q)$ of the FK percolation [Gri95]. As a special case of our inequality, we obtain a "strong" version of the monotonicity in p (see (1.1) below), which quantifies how much stochastically larger $\mathrm{FK}_{\tilde{p},q}$ is compared to $\mathrm{FK}_{p,q}$ for parameters $\tilde{p} > p$.

During the preparation of the paper we discovered that a subcase of (1.3) has already been shown in [Sev24], see Lemma 2.5.

The results of this paper therefore sit between two standard approaches. On the one hand, classical correlation inequalities such as FKG [FKG71, Hol74] and the Ahlswede-Daykin four-function theorem [AD78], GKS inequalities [Gri67, KS68], and GHS [GHS70] give robust monotonicity statements for spins, correlations, and related observables. On the other hand, modern coupling methods compare dependent fields with Bernoulli-type percolations, often relying on [LSS97] (see e.g. [DG25] in the context of the XY model or [DM24] in the context of random geometry). Other methods have been fruitfully invented as well to get finer results, e.g. [BFO25].

Here the full edge-energy field is kept, but the order relation is relaxed by an explicit independent thinning. The main results identify the exact amount of thinning needed, and the counterexamples show that these thresholds cannot be improved.

A particular case of this new stochastic domination is used to prove part of the results about weak monotonicity of the energy field of the Ising model, thus relating the two parts of the paper. We start by explaining our results on FK percolation.

1.1. **Strong monotonicity on FK percolation.** Let $G = (V, E)$ be a finite graph. We start by defining the FK percolation on G, introduced for the first time in [FK72] and which generalizes Bernoulli percolation and depends on an additional global parameter $q > 0$. For this, we view a percolation configuration $\omega \in \{0,1\}^E$ equivalently as a vector indexed by E or as the subset of open edges $\{e \in E : \omega_e = 1\}$. Letting $k(\omega)$ denote the number of connected components in the subgraph induced by ω, the FK percolation model is defined as the probability measure on $\{0,1\}^E$ given by

$$\mathrm{FK}_{p,q}(\omega) \propto q^{k(\omega)} \prod_{e \in E} p_e^{\omega_e}(1-p_e)^{1-\omega_e}.$$

In this paper, we restrict our attention to the regime $q \geq 1$; for $q < 1$, the model's behavior changes fundamentally, and the results proven here no longer hold. Furthermore, for the special case $q = 1$, we denote $\mathrm{FK}_{p,1} = \mathrm{Ber}_p$ the Bernoulli percolation.

For $q \geq 1$, it is a well-known result that $\mathrm{FK}_{p,q}$ is stochastically increasing in the parameter p and stochastically decreasing in the parameter q (see, e.g., Theorem 3.21 in [Gri06]). Our first result generalizes and strengthens both of these monotonicities.

Theorem 1.1. *Let $q \geq 1$ and let $p, \tilde{p} \in [0,1]^E$ be parameters such that $p \leq \tilde{p}$ pointwise. We have the following stochastic domination:*

$$\mathrm{FK}_{p,q} \preccurlyeq \mathrm{FK}_{\tilde{p},q} \cap \mathrm{Ber}_{p/\tilde{p}}, \qquad \text{(Strong stochastic domination)} \tag{1.1}$$

where, for two percolation measures μ and ν, $\mu \cap \nu$ denotes the law of $\omega \cap \omega'$ for independent configurations $\omega \sim \mu$ and $\omega' \sim \nu$. The ratio $p/\tilde{p}$ is understood edgewise.

More generally, for any real numbers $q, \tilde{q} \geq 1$ and parameters $p, \tilde{p} \in [0,1]^E$, we have

$$\mathrm{FK}_{p\tilde{p},q\tilde{q}} \preccurlyeq \mathrm{FK}_{p,q} \cap \mathrm{FK}_{\tilde{p},\tilde{q}}, \qquad \textit{and} \tag{1.2}$$

$$\mathrm{FK}_{\widehat{p},q\tilde{q}} \succcurlyeq \mathrm{FK}_{p,q} \cup \mathrm{FK}_{\tilde{p},\tilde{q}}, \tag{1.3}$$

where

$$\widehat{p} := (p^*\tilde{p}^*)^* = \frac{p\tilde{p} + p\tilde{q}(1-\tilde{p}) + \tilde{p}q(1-p)}{(1-p)(1-\tilde{p}) + p\tilde{p} + p\tilde{q}(1-\tilde{p}) + \tilde{p}q(1-p)}. \tag{1.4}$$

(The reason for the notation $(p^*\tilde{p}^*)^*$ will be made clear in Remark 2.8.)

Although the proof of this theorem relies only on a previously known argument, the Holley inequality (see Theorem 2.3 below), the result did not appear before. A weaker form of (1.3) for the specific case of $\tilde{q} = 1$ was recently obtained by Severo in the context of slab percolation for the Ising model (see Lemma 2.5 in [Sev24]), using instead the Russo formula for FK percolation. Furthermore, as discussed in Remark 2.5, the bounds provided by Theorem 1.1 are tight for all $q \geq 1$ in both the high-density $p \nearrow 1$ and low-density $p \searrow 0$ regimes.

1.2. **Weak monotonicities on the energy field.** For a spin configuration $\sigma \in \{\pm 1\}^V$, we are primarily interested in the percolation configuration $\xi = \xi(\sigma)$, defined as the set of edges $e = (xy)$ such that $\sigma_x = \sigma_y$. More precisely,

$$\xi := \{e = (xy) \in E : \sigma_x = \sigma_y\} \quad \text{or equivalently} \quad \xi_{xy} := \mathbf{1}_{\sigma(x)=\sigma(y)} \quad \forall x \sim y.$$

When σ has the law of an Ising model with coupling constants J, we say that $\xi(\sigma) \sim \mathcal{E}_J$ follows the law of an Ising energy field on G with coupling constants J. The energy field is a commonly studied observable of the Ising model itself, modern works have focused on planar domains for the critical or near critical parameters [HS13, IKT24, GK25]. In this work, we are more interested in the behaviour on a general graph.

The relationship between the energy field with FK percolation is given by the Edwards–Sokal coupling (see [ES88], or Theorem 1.13(b) in [Gri06] for a more modern approach). In our notation it says that

$$\mathrm{FK}_{p,2} = \mathcal{E}_J \cap \mathrm{Ber}_p, \tag{1.5}$$

whenever $p_e = p(J)_e = 1 - e^{-2J_e}$ for every edge $e \in E$. Taking $q = 2$ in Theorem 1.1 therefore gives the first weak monotonicity theorem for the Ising energy field.

Theorem 1.2 (Weak stochastic monotonicity for the Ising energy field)**.** *Let $J \leq \tilde{J}$ be (edge-dependent) coupling constants, where the inequality holds pointwise for every edge. Then, for any parameter $p \leq 1 - \exp(-2J)$,*

$$\mathcal{E}_J \cap \mathrm{Ber}_p \preccurlyeq \mathcal{E}_{\tilde{J}} \cap \mathrm{Ber}_p. \tag{1.6}$$

Moreover, one cannot take $p > 1 - \exp(-2J)$ in general.

We refer to the type of inequality in (1.6) as *weak domination*, the general definition and its basic properties are given below and again in Section 3.1.1. We are primarily concerned with the optimal parameter p that can be chosen in the above inequality. As states the last sentence of Theorem 1.2, if one requires this inequality to hold in general, the maximum possible parameter p for which the domination holds is $1 - e^{-2J}$ (for a precise and rigorous formulation of this optimality, see Theorem 4.1 and Remark 4.2). The domination (1.6) itself, however, swiftly follows from the Edwards-sokal coupling and Theorem 1.1:

Proof of (1.6). For $p = p(J)$, the left-hand side is exactly $\mathrm{FK}_{p(J),2}$ by equation (1.5). Similarly, by the Edwards-sokal coupling, the right-hand side can be decomposed as

$$\mathcal{E}_{\tilde{J}} \cap \mathrm{Ber}_{p(\tilde{J})} \cap \mathrm{Ber}_{p(J)/p(\tilde{J})} = \mathrm{FK}_{p(\tilde{J}),2} \cap \mathrm{Ber}_{p(J)/p(\tilde{J})}.$$

The stochastic domination is then a direct application of (1.1). The fact that it holds for smaller values of p follows trivially by intersecting both sides with a further independent Bernoulli percolation of parameter $p/p(J)$. □

Note that Theorem 1.1 is essential to obtain (1.6): the classical stochastic monotonicity of $\mathrm{FK}_{p,2}$ in the parameter p would instead rewrite as a stochastic domination where the two sides are intersected with Bernoulli percolations of *different* parameters. Our strengthening of the FK-percolation monotonicity enables the use of the *same parameter* p on both sides.

Remark 1.3. A completely analogous result to (1.6) holds true for the q–Potts model for any integer $q \geq 2$, where the definition of the energy field and the proof are analogous to the case $q = 2$. However, in this paper we concentrate on the Ising model, as our subsequent results are based on techniques developed specifically for it. Moreover, the energy field is especially interesting for the Ising model but far less so for larger values of q, since on a connected graph the spin configuration (up to a global flip) is measurable with respect to its energy field, a fact that is not true for the Potts model when $q \geq 3$.

Theorem 1.2 opens the door to a general study of weakened stochastic orders on probability measures on $\{0,1\}^E$. Weak domination asks for stochastic domination after a common independent thinning by a positive parameter. This turns out to be slightly stronger than the inequalities $\mu(A) \leq \nu(A)$ for A running on a specific subset of increasing events. See Section 3 and Proposition 3.1.

One may then define an in-between notion of stochastic domination. We say that μ *is weakly*† *dominated by* ν, written $\mu \preccurlyeq^\dagger \nu$, if at the same time μ is weakly stochastically dominated by ν (i.e. for some parameter $p > 0$ the domination $\mu \cap \mathrm{Ber}_p \preccurlyeq \nu \cap \mathrm{Ber}_p$ holds), and for some parameter $q < 1$ the domination $\mu \cup \mathrm{Ber}_q \preccurlyeq \nu \cup \mathrm{Ber}_q$ holds as well. In Section 3.1.2 we define again this notion and we give a dual interpretation of the domination $\mu \cup \mathrm{Ber}_q \preccurlyeq \nu \cup \mathrm{Ber}_q$.

Although in this paper we apply the notions of weak and weak† domination only to the energy field, the general framework developed in Section 3 is one of the model-independent components of the paper. It isolates a way of quantifying how much stochastic order remains after classical domination has failed: weak domination compares the two measures after a common Bernoulli thinning, while weak† domination also imposes the dual comparison, obtained through a common Bernoulli sprinkling. We believe that these notions, and their analogues for FKG, may be useful in other contexts where exact monotonicity or positive association is absent, but where the models remain close to satisfying such properties.

In the following theorem, we find the optimal threshold for weak† domination of the energy field of the Ising model.

Theorem 1.4 (Threshold for weak† domination)**.** *Let $J \leq \tilde{J}$ be (edge-dependent) coupling constants, where the inequality holds pointwise. Then, for any parameter $p \in [0,1]^E$ with $p < p^\dagger(J) := 1 - \exp(-4J)$ pointwise, we have*

$$\mathcal{E}_J \cap \mathrm{Ber}_p \preccurlyeq^\dagger \mathcal{E}_{\tilde{J}} \cap \mathrm{Ber}_p. \tag{1.7}$$

Furthermore, for $p \geq p^\dagger(J)$ the result does not hold in general.

As before, the last sentence of Theorem 1.4 summarizes a sharpness statement. The precise sense for which $p^\dagger$ is the optimal threshold parameter is detailed in Theorem 4.5, as explained in Subsection 4.3. Note that in general one cannot take $p = p^\dagger(J)$ in (1.7). This is not a contradiction since the weak† domination is not closed under weak limits of measures.

To keep the introduction focused, we do not state the parallel results for the FKG property here. However in Theorems 4.1 and 4.5, we obtain not only the results for weak and weak† stochastic domination, but also for the corresponding weak and weak† FKG properties for the Ising energy field.

1.3. **Ideas of the proofs and organization.** The proofs have two separate components. The FK comparison Theorem 1.1 is proved first, using Holley's inequality. Combined with the Edwards–Sokal identity, it gives the weak monotonicity result at the parameter $p(J) = 1 - e^{-2J}$, as explained above.

The weak† threshold $p^\dagger(J) = 1 - e^{-4J}$ is obtained by a different argument. We introduce and study the model $\mathcal{E}_J^\dagger := \mathcal{E}_J \cap \mathrm{Ber}_{1-e^{-4J}}$ and prove, in Proposition 4.16, inequalities that are a slightly weaker form of weak† domination at the threshold $p^\dagger(J)$. The proof uses a high-temperature expansion for an abstract Ising model associated with the cycle space of G. On planar graphs this is the usual dual high-temperature expansion; on general finite graphs it replaces the missing planar dual by an abstract Ising model (see Remark 4.10 for a discussion on why this makes sense). This is the first step and the main difficulty to achieve weak† domination (and the parallel notion of weak† FKG) below $p^\dagger$. A technical lemma then turns these endpoint inequalities into weak† domination, and weak† FKG, for every strictly smaller parameter.

The introduction of an abstract duality is also close in spirit of the argument we highlight in Remark 2.8 to explain why one can see that the two statements of Theorem 1.1 about FK percolation are equivalent.

Optimality is proved by explicit counterexamples. For weak† domination, a cycle with four vertices already shows that parameters larger than $p^\dagger(J)$ cannot be allowed in general. For weak domination above the Edwards–Sokal threshold, the counterexample is chosen to be a cycle as well, though it needs to be quite large. The argument uses a decomposition of the energy field into a linear combination of Bernoulli measures, including one with a parameter outside $[0, 1]$, which makes it possible to detect the failure of monotonicity for a suitable increasing event.

The paper is organized as follows. Section 2 proves the FK comparison Theorem 1.1. Section 3 introduces weak domination, weak† domination, and the corresponding weak FKG notions, together with the general facts used later. Section 4 proves the Ising results, develops the abstract high-temperature expansion for $\mathcal{E}_J^\dagger$, and gives the counterexamples establishing optimality of the two thresholds.

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2. A new inequality for the FK-model

The objective of this section is to prove Theorem 1.1. That is to say to show that the intersection of two independent FK-models stochastically dominates the FK-model whose parameters are the products of the former ones.

Let us start by recalling the definition of the FK-model. Fix $G = (V, E)$ a finite graph, a real number $q > 0$ and $p \in [0, 1]^E$ an assignation to each edge of a number in $[0, 1]$. The FK-model (with free boundary conditions) is the probability measure on percolation configurations $\omega \in P(E) = \{0, 1\}^E$

$$\mathrm{FK}_{p,q}(\omega) \propto \prod_{e\in\omega} p_e \prod_{e\in E\setminus\omega} (1 - p_e)\; q^{k(\omega)},$$

where $k(\omega)$ is the amount of connected components of the graph $G_\omega = (V, \omega)$. In the following, we are always making the identification between ω as a subset of E or ω as an element of $\{0, 1\}^E$, in this context for an edge $e \in E$, the proposition $e \in \omega$ is equivalent to $\omega(e) = 1$.

We can now be precise with the two main results of this section. To do this, recall that for any μ, ν two probability measures on $P(E)$, we define $\mu \cap \nu$ as the push-forward under the intersection of the measure $\mu \times \nu$. In words, it is the law of the intersection of two independent copies of $\omega_1 \sim \mu$ with $\omega_2 \sim \nu$.

Theorem 2.1. *For any parameters $p, \tilde{p} \in [0,1]^E$ and real numbers $q, \tilde{q} \geq 1$, we have the following stochastic domination:*

$$\mathrm{FK}_{p,q} \cap \mathrm{FK}_{\tilde{p},\tilde{q}} \succcurlyeq \mathrm{FK}_{p\tilde{p},q\tilde{q}}. \tag{2.1}$$

Defining the union $\mu \cup \nu$ of two probability measures μ, ν in the same way, one gets a complementary (and in fact equivalent, see Remark 2.8) stochastic domination for the union of two FK measures.

Theorem 2.2. *For any parameters $p, \tilde{p} \in [0,1]^E$ and real numbers $q, \tilde{q} \geq 1$, we have the following stochastic domination:*

$$\mathrm{FK}_{p,q} \cup \mathrm{FK}_{\tilde{p},\tilde{q}} \preccurlyeq \mathrm{FK}_{\widehat{p},q\tilde{q}} \tag{2.2}$$

where $\widehat{p}$ is given by the formula (1.4).

The proofs of these two theorems are very similar and based on the use of Holley inequality, that we recall below. First, we need to introduce the following numbers. Let μ be a probability measure on $\{0,1\}^E$, $e \in E$ an edge and $\xi \in \{0,1\}^{E\setminus\{e\}}$ a percolation configuration outside of e, and define

$$\mu(e \mid \xi) := \mu\left(e \in \omega \;\middle|\; \omega|_{E\setminus\{e\}} = \xi\right).$$

This is the probability that a given edge belongs to ω given the sample outside the edge. The following, known as Holley inequality, has proven to be very useful to show stochastic dominations in many contexts. Though we don't give proof, let us say that it can be deduced quickly from a coupling of two Markov chains.

Theorem 2.3 (Holley inequality, [Hol74])**.** *Let μ, ν measures of full support on $\{0,1\}^E$. Let us assume that for any $e \in E$ and $\xi, \xi' \in \{0,1\}^{E\setminus\{e\}}$ with $\xi' \leq \xi$, we have the following inequality*

$$\mu(e \mid \xi') \leq \nu(e \mid \xi). \tag{2.3}$$

Then $\mu \preccurlyeq \nu$.

In order to make use of this criteria for stochastic domination, we want to know what are the values $\mu(e, \xi)$ for μ a FK-percolation measure. The answer is given by the following proposition:

Proposition 2.4 (One-point function conditioned to the exterior, Theorem 3.1 in [Gri06])**.** *For $q > 0$ and $p \in [0,1]^E$, the one point function of the FK-model conditionnally on the outside is given by*

$$\mathrm{FK}_{p,q}(e \mid \xi) = \begin{cases} p_e & \text{if } \xi \text{ connects the two endpoints of } e, \\ \frac{p_e}{p_e + q(1-p_e)} & \text{if } \xi \text{ does not.} \end{cases} \tag{2.4}$$

Furthermore, this property defines $\mathrm{FK}_{p,q}$.

Remark 2.5. Let us now use Proposition 2.4 to see that Theorem 2.1 is sharp at least when both $p, \tilde{p} \searrow 0$ or both $p, \tilde{p} \nearrow 1$. For the first case note that when $p \searrow 0$, w.h.p. $\omega|_{\{e\}^c} = 0$ and therefore

$$\mathrm{FK}_{p,q}(\omega_e) \sim \frac{p_e}{p_e + (1-p_e)q}(1 + o(1)) \sim \frac{p_e}{q},$$

which achieves asymptotical equality for (2.1).

For the close to 1 case, take $p \nearrow 1$, and note that w.h.p. $\omega_{\{e\}^c} = 1$, and therefore

$$\mathrm{FK}_{p,q}(\omega_e) \sim p_e(1 + o(1)) \sim p_e,$$

which also achieves which achieves asymptotical equality for (2.1). A similar reasoning can be done to show the sharpness of Theorem 2.2.

We now have all the tools to prove Theorems 2.1 and 2.2. The two proofs are the same though the computations are a bit more tedious for the proof of Theorem 2.2, as one can guess from the form of $\widehat{p}$. Since we do not use Theorem 2.2, the union statement, in the rest of the paper, we chose to first prove the intersection statement carefully, Theorem 2.1, and then explain how to adapt the proof for Theorem 2.2.

We explain in Remark 2.8 how one can see that the two statements are actually equivalent, and give a sense of why $\widehat{p}$ is given by the complicated expression (1.4). We actually use this viewpoint to smoothen some of the computations not performed in the proof of Theorem 2.2.

Proof of Theorem 2.1. We will show that the measures $\mu := \mathrm{FK}_{p\tilde{p},q\tilde{q}}$ and $\nu := \mathrm{FK}_{p,q} \cap \mathrm{FK}_{\tilde{p},\tilde{q}}$ satisfy together the assumptions of Theorem 2.3, which is of course enough. We may assume that the parameters $p, \tilde{p}$ are strictly between 0 and 1, those limiting cases can be seen as weak limits of the general case. Therefore, the distributions μ, ν are of full support, so we focus on (2.3).

Let us first note that for all $q \geq 1$ and $p \in [0,1]^E$, the function $\mathrm{FK}_{p,q}(e \mid \cdot)$ is nondecreasing in the outside percolation ξ. This follows from the fact that the event that e^+ is connected to e^- using ξ is increasing on ξ and that $p_e \geq p_e/(p_e + q(1-p_e))$, as $q \geq 1$.

From the monotonicity of the function $\mathrm{FK}_{p,q}(e \mid \cdot)$, we note that it is enough to show Holley inequality (2.3) restricted to the case $\xi = \xi'$. That is to say that for any $e \in E$ and $\xi \in \{0,1\}^{E\setminus\{e\}}$

$$\nu(e \mid \xi) \geq \mu(e \mid \xi) = \mathrm{FK}_{p\tilde{p},q\tilde{q}}(e \mid \xi). \tag{2.5}$$

To do this, let us lower bound the left hand side.

Claim 2.6. *If ξ connects the two endpoints of e, $\nu(e \mid \xi) = p\tilde{p}$. If ξ does not connect the two endpoints of e, $\nu(e \mid \xi) \geq \frac{p}{p+q(1-p)} \frac{\tilde{p}}{\tilde{p}+\tilde{q}(1-\tilde{p})}$.*

We now assume the claim and show (2.5). In this case, it is enough to show that

$$p\tilde{p} \geq p\tilde{p} \qquad \text{and} \qquad \frac{p}{p+q(1-p)} \frac{\tilde{p}}{\tilde{p}+\tilde{q}(1-\tilde{p})} \geq \frac{p\tilde{p}}{p\tilde{p}+q\tilde{q}(1-p\tilde{p})}. \tag{2.6}$$

Of course, the first inequality is trivial; the second can be shown to be equivalent to

$$q(1-p)(\tilde{q}-1)\tilde{p} + (q-1)p\tilde{q}(1-\tilde{p}) \geq 0,$$

which is true for values of $p_i \in [0,1]^E$ and $q \geq 1$. □

We are just left with the proof of the claim.

Proof of Claim 2.6. Let us denote $\mathbb{E}_\xi$ the expectation with respect to $\mathrm{FK}_{p,q} \otimes \mathrm{FK}_{\tilde{p},\tilde{q}}$ conditioned on the event

$$\left\{ (\omega \cap \tilde{\omega})|_{\{e\}^c} = \xi \right\}.$$

By definition, $\nu(e \mid \xi) = \mathbb{E}_\xi[\omega_e = \tilde{\omega}_e = 1]$, therefore

$$\nu(e \mid \xi) = \mathbb{E}_\xi \Big[\mathrm{FK}_{p,q}(e \mid \omega|_{\{e\}^c}) \mathrm{FK}_{\tilde{p},\tilde{q}}(e \mid \tilde{\omega}|_{\{e\}^c}) \Big]. \tag{2.7}$$

However, since $q, \tilde{q} \geq 1$,

$$\mathrm{FK}_{p,q}(e \mid \omega|_{\{e\}^c}) \geq \frac{p}{p+q(1-p)} \qquad a.s. \tag{2.8}$$

$$\mathrm{FK}_{\tilde{p},\tilde{q}}(e, \tilde{\omega}|_{\{e\}^c}) \geq \frac{\tilde{p}}{\tilde{p}+\tilde{q}(1-\tilde{p})} \qquad a.s. \tag{2.9}$$

and therefore the right hand side is at least $\frac{p}{p+q(1-p)} \frac{\tilde{p}}{\tilde{p}+\tilde{q}(1-\tilde{p})}$. However, in the case that ξ connects the endpoints of e, we have that, $\mathbb{E}_\xi$–a.s., $\omega|_{\{e\}^c}, \tilde{\omega}|_{\{e\}^c}$ both connect as well the endpoints of e. Therefore, in this case,

$$\mathrm{FK}_{p,q}(e \mid \omega|_{\{e\}^c}) = p \qquad a.s. \tag{2.10}$$

$$\mathrm{FK}_{\tilde{p},\tilde{q}}(e, \tilde{\omega}|_{\{e\}^c}) = \tilde{p} \qquad a.s. \tag{2.11}$$

whence the claim. □

Proof of Theorem 2.2. The proof of the union statement follows closely the proof presented above for (2.1). We define $\mu = \mathrm{FK}_{p,q} \cup \mathrm{FK}_{\tilde{p},\tilde{q}}$ and $\nu = \mathrm{FK}_{\widehat{p},q\tilde{q}}$ and we use Holley Theorem 2.3 to the pair μ, ν. Of course, as in the proof of the intersection statement, we may assume that $p, \tilde{p}$ are always strictly between 0 and 1, so that the meaures μ, ν are of full support. We may thus focus on proving Holley inequality (2.3).

The relevant claim is then

Claim 2.7. *Let $e \in E$ be any edge, and ξ be a configuration on the set $E \setminus \{e\}$. If ξ does not connect the two endpoints of e, $\mu(e \mid \xi) = \nu(e \mid \xi)$. If ξ does connect the two endpoints of e, $\mu(e \mid \xi) \leq p + \tilde{p} - p\tilde{p}$.*

This claim is proven similarly to Claim 2.6: for the first point, one must have $1 - \mu(e \mid \xi) = (1 - \frac{p}{p+q(1-p)})(1 - \frac{\tilde{p}}{\tilde{p}+\tilde{q}(1-\tilde{p})})$ so the equality follows from $\nu(e \mid \xi) = \frac{\widehat{p}}{\widehat{p}+q\tilde{q}(1-\widehat{p})}$ and the additional relation

$$\left(1 - \frac{p}{p+q(1-p)}\right) \cdot \left(1 - \frac{\tilde{p}}{\tilde{p}+\tilde{q}(1-\tilde{p})}\right) = 1 - \frac{\widehat{p}}{\widehat{p}+q\tilde{q}(1-\widehat{p})}. \tag{2.12}$$

(This equality is smoothly proved in Remark 2.9 using the relation $\widehat{p} = (p^*\tilde{p}^*)^*$ explained in Remark 2.8.)

For the second point, the same reasoning of the proof of Claim 2.6 gives $1 - \mu(e \mid \xi) \geq (1-p)(1-\tilde{p})$, that is $\mu(e \mid \xi) \leq p + \tilde{p} - p\tilde{p}$. This shows the claim.

Furthermore, this claim can be used to prove (2.5) since in the case $e^- \overset{\xi}{\not\leftrightarrow} e^+$ we have $\mu(e \mid \xi) = \nu(e \mid \xi)$ and in the case $e^- \overset{\xi}{\leftrightarrow} e^+$ we have $\mu(e \mid \xi) \leq \nu(e \mid \xi)$ since

$$p + \tilde{p} - p\tilde{p} \leq \widehat{p}.$$

(Once again, we show this last inequality in a meaningful way in Remark 2.9.)

Hence, if $e \in E$ is an edge, and $\xi' \leq \xi$ are configurations on the set $E \setminus \{e\}$, then $\mu(e \mid \xi') \leq \nu(e \mid \xi') \leq \nu(e \mid \xi)$ (the second inequality relies on the monotonicity of $\nu(e \mid \cdot)$ since ν is a FK measure) so that Holley stochastic domination criteria applies: we have the stochastic domination (2.2). □

Remark 2.8. It can actually be seen that the two stochastic dominations are equivalent in a way that explains the reason why we wrote $\widehat{p}$ in the form $(p^*\tilde{p}^*)^*$. Indeed, in the case of G a planar graph, if $p^* := \frac{q(1-p)}{p+q(1-p)}$, $\tilde{p}^* := \frac{\tilde{q}(1-\tilde{p})}{\tilde{p}+\tilde{q}(1-\tilde{p})}$ and $(p^*\tilde{p}^*)^* := \frac{q\tilde{q}(1-p^*\tilde{p}^*)}{p^*\tilde{p}^*+q\tilde{q}(1-p^*\tilde{p}^*)} = \widehat{p}$ then by the standard fact that the dual of $\mathrm{FK}_{p,q}$ percolation is $\mathrm{FK}_{p^*,q}$ percolation on the dual graph with dual parameter $p^* = \frac{q(1-p)}{p+q(1-p)}$ (see e.g. Section 6.1. in [Gri06]), the domination (2.2) is precisely the domination (2.1) reversed where we take the dual edges. Note that the map $\cdot \mapsto \cdot^*$ *depends on the parameter* q, and that in the formula $\widehat{p} = (p^*\tilde{p}^*)^*$, we actually use three such maps, namely for the three parameters $q, \tilde{q}, q\tilde{q}$.

It is actually possible in the case of a general graph G to define a somehow more general FK percolation for which Theorem 2.1 still holds (with precisely the same proof) and which allows to take the complementary even in the absence of a dual graph, so that the equivalence between Theorem 2.1 and 2.2 is always true. We do not pursue this further, but a similar theory is developed for the Ising model in Subsection 4.4.1, where we introduce an abstract Ising model allowing some duality even in the case of a non-planar graph.

Remark 2.9. Let us explain why (2.12) holds, without simply plugging the value given by (1.4). Indeed, in the notations of the previous remark, it becomes $p^*\tilde{p}^* = (\widehat{p})^*$ (where the * are respectively for the parameters $q, \tilde{q}, q\tilde{q}$), which is precisely our definition of $\widehat{p}$ (since taking the dual parameter is an involution whatever the q–parameter is).

Now, let us explain where $p+\tilde{p}-p\tilde{p} \leq \widehat{p}$ comes from. First, rewrite it as $(1-p)(1-\tilde{p}) \geq 1-\widehat{p}$. Second, since p, p^* are related by $1 - p = \frac{p^*}{p^*+q(1-p^*)}$ (and similarly for $\tilde{p}, \tilde{p}^*$ and $\widehat{p}, p^*\tilde{p}^*$), the inequality is precisely the right hand side of (2.6) but with $p, \tilde{p}$ replaced by $p^*, \tilde{p}^*$, and so is true by the same rearrangement.

These two justifications once again show that what we really use is (the proof of) the intersection statement for the dual parameters, as explained in the previous remark.

3. Variants of stochastic domination and FKG inequality

Motivated by the fact that the energy field of Ising is not stochastically monotone with respect to the temperature, we introduce, in this section, two different versions of stochastic domination. We will see afterwards, in Section 4 that, in fact, the energy field of Ising model is monotone in the temperature under these weaker senses in a very precise way.

The main idea of these weaker variants of stochastic domination is to hide the anomalies that do not allow the domination via the intersection with an independent Bernoulli percolation. Thus, to simplify the notation, for any probability measure μ and parameter $p \in [0,1]^E$, we write $\mu_p = \mu \cap \mathrm{Ber}_p$. We denote as well μ^c the law of ω^c where $\omega \sim \mu$ and we call it the *dual measure* of μ.

Additionally, in this section some specific events play a key role. For $F \subset E$, we denote $\forall_F$, and A_F, the event that all edges $e \in F$ are open and closed respectively for the percolation configuration $\omega \in \{0,1\}^E$. More precisely

$$\forall_F := \{\omega \mid F \subset \omega\} \quad \text{and} \quad \mathsf{A}_F = \{\omega \mid \omega \cap F = \emptyset\}.$$

The notation $\forall_F$ comes from the fact that we ask *all* edges in F to be open, and the notation A is simply the $\pi-$rotation of $\forall$. Keeping in mind that the events $\forall_F$ and A_F are somehow dual of each other, one sees that

$$\mu^c(\mathsf{A}_F) = \mu(\forall_F). \tag{3.1}$$

We start this section by introducing and discussing two variants of stochastic dominations. Then, we discuss the weakening of the FKG properties in this context. We end by discussing a natural strengthening of the stochastic domination.

3.1. **Weakening of stochastic domination.** As stated before, we introduce two notions of stochastic domination. The weaker one called *weak stochastic domination* and a stronger one called *weak*† *stochastic domination.*

3.1.1. *Weak stochastic domination.* Let us start by introducing the weakest form of stochastic domination.

Definition 1. *Take μ, ν two probability measures on $\{0,1\}^E$ and $p \in (0,1]^E$. We say that the measure μ is p–dominated by ν if $\mu_p \preccurlyeq \nu_p$. We denote this relationship $\mu \preccurlyeq_p \nu$. Furthermore, we say that μ is weakly dominated by ν if there exists a parameter $p \in (0,1]^E$ such that μ is $p-$dominated by ν.*

As a first observation of this new notion, let us note that if two probability measures μ, ν on $0,1^E$ are stochastically ordered, that is $\mu \preccurlyeq \nu$, then for any parameter $p \in (0,1]^E$ one also has $\mu_p \preccurlyeq \nu_p$. The converse, however, fails in general: the existence of a parameter p with strictly positive coordinates such that $\mu_p \preccurlyeq \nu_p$ does *not* imply that $\mu \preccurlyeq \nu$[1]. Consequently, this notion is strictly weaker than classical stochastic domination, and gets weaker and weaker as p becomes smaller.

The reason we are interested in this definition is because we want to find the best parameter p for which the energy field of Ising (defined in the Introduction and in Section 4) is p–monotone, in other words the largest p such that $\mathcal{E}_{J,p}$ is monotone in J.

We now list elementary properties of the weak stochastic domination.

Proposition 3.1. *Let μ, ν be probability measures on $\{0,1\}^E$.*

(i) if $p \leq \tilde{p}$ are two parameters, and $\mu \preccurlyeq_{\tilde{p}} \nu$, then $\mu \preccurlyeq_p \nu$.
(ii) if μ is weakly dominated by ν, then $\mu(\forall_F) \leq \nu(\forall_F)$ for all $F \subset E$.
(iii) if for any $F \subset E$, $\mu(\forall_F) \leq \nu(\forall_F)$, and the inequality is strict for any nonempty F, then μ is weakly dominated by ν.

Proof. First, note that (i) follows as $\mu_p = \mu_{\tilde{p}} \cap \mathrm{Ber}_{p/\tilde{p}}$ and the same is true for ν_p, whence the result. To prove (ii), we use that $\mu_p(\forall_F) = \mu(\forall_F) \prod_{e \in F} p_e$ and that the analogue is true for ν_p. The result follows upon by dividing by $\prod_{e \in F} p_e > 0$.

Finally, to prove (iii), we need to find $p \in (0,1)^E$ such that for any increasing event A, $\mu_p(A) \leq \nu_p(A)$. As $\mathcal{P}(E)$ is finite, it is enough to show that for all increasing sets A there is $p > 0$ and constant on each edge such that $\mu_p(A) \leq \nu_p(A)$.

In case that A is either $\forall_\emptyset$ or $\forall_E$, the conclusion is part of the assumption. In the other cases, A can be written as

$$A = \bigcup_F \forall_F$$

[1]An explicit counterexample is provided e.g. in the proof of Theorem 1.2, Subsection 4.2

where F runs on the set of minimal (for inclusion) witnesses of A. Define $m = m(A)$ as the minimal number of edges that such a witness can have. One can see that as $p \searrow 0$

$$\mu_p(A) = p^m \sum_F \mu(\forall_F) + o(p^m) \tag{3.2}$$

where the sum runs over all witnesses F of cardinal m. A similar decomposition holds for $\nu_p(A)$ and by assumption, since the first non-zero order term is strictly smaller for μ than for ν, one has $\mu_p(A) < \nu_p(A)$ for all $p > 0$ sufficiently small. □

Remark 3.2. If the inequalities $\mu(\forall_F) \leq \nu(\forall_F)$ in the third item are not required to be strict, the implication of weak domination is actually false as shown by the following example: Take $E = \{1, 2\}$, $\mu = \mathrm{Ber}_{1/2}$ and $\nu = \frac{1}{2}\delta_{(0,0)} + \frac{1}{2}\delta_{(1,1)}$, the measure where one tosses one fair coin to decide if all edges are open or close. One can check that $\mu(\forall_F) \leq \nu(\forall_F)$ for any $F \subseteq E$, but for every positive parameter p, $\mu_p(A) > \nu_p(A)$ where A is the increasing event $\{$at least one edge is open$\}$.

3.1.2. *Weak*† *stochastic domination.* In this section, we introduce a notion of stochastic domination that lies in between the weak and the classical stochastic domination. To do this, it is useful to introduce the notion of dual weak stochastic domination.

Definition 2 (Dual weak stochastic domination)**.** *For μ, ν probability measures on $\{0, 1\}^E$ and a parameter $q \in [0, 1)^E$, we say that ν dually q–weakly dominates μ, relation denoted $\mu \preccurlyeq^c_q \nu$, if $\nu^c \preccurlyeq_{1-q} \mu^c$. If there exists such a parameter q[2], we say that ν dually weakly dominates μ.*

Note that this dual notion of domination may be understood as follows. The fact that $\mu \preccurlyeq^c_q \nu$ is equivalent to the domination $\mu \cup \mathrm{Ber}_q \preccurlyeq \nu \cup \mathrm{Ber}_q$. Note that the second point of Proposition 3.1 implies the inequalities $\mu(\mathsf{A}_F) \geq \nu(\mathsf{A}_F)$ for all $F \subset E$, and the third point implies that in case of strict inequalities for all $\emptyset \neq F \subset E$, we have dual weak domination.

We can now define the concept of weak† stochastic domination.

Definition 3 (Weak† stochastic domination)**.** *If μ, ν are two probability measures on $\{0, 1\}^E$, we say that ν weakly† dominates μ if there are parameters $p \in (0, 1]^E$ and $q \in [0, 1)^E$ such that $\mu \preccurlyeq_p \nu$ and $\mu \preccurlyeq^c_q \nu$. We denote this relation $\mu \preccurlyeq^\dagger \nu$, and we add p and q as sub-indices if needed.*

The reason why we introduce this concept is the following: returning to the situation where we do have a weak domination between measures μ, ν and we are interested in the largest parameter for which stochastic domination holds, we can also investigate the largest parameter p for which $\mu_p \preccurlyeq^\dagger \nu_p$, a weaker property, that might make arise a different, and interesting, new threshold. This is the case for μ, ν the energy fields of Ising at different temperatures as we will see in Section 4.

Let us now check[3] that the weak† domination also satisfies the first point of Proposition 3.1.

Proposition 3.3. *Let μ, ν be probability measures on $\{0, 1\}^E$ such that $\mu \preccurlyeq^\dagger_{p,q} \nu$. Then, for any parameter $r \in (0, 1]^E$, $\mu_r \preccurlyeq^\dagger_{p',q'} \nu_r$, with p', q' defined by $p' = 1 \wedge (p/r)$ and $q' = qr'$. Here, $r' = r/(1 - q(1 - r))$.*

In particular, weak† *domination is stable under intersection with* Ber_r.

Proof. First, we see that it is direct that $\mu \preccurlyeq_{p'r} \nu$, i.e. $\mu_r \preccurlyeq_{p'} \nu_r$ by monotonicity (first bullet point of Proposition 3.1, since $p'r \leq p$). To show that $\mu_r \preccurlyeq^c_{q'} \nu_r$, we need the following claim.

Claim 3.4. *With the notations of the proposition. Then, we have the following equality of measures:*

$$\big(\mu \cap \mathrm{Ber}_r\big) \cup \mathrm{Ber}_{q'} = \big(\mu \cup \mathrm{Ber}_q\big) \cap \mathrm{Ber}_{r'}. \tag{3.3}$$

Let us first see how the claim implies that $\mu_r \preccurlyeq^c_{q'} \nu_r$. This follows from computing

$$\mu_r \cup \mathrm{Ber}_{q'} = (\mu \cup \mathrm{Ber}_q) \cap \mathrm{Ber}_{r'} \preccurlyeq (\nu \cup \mathrm{Ber}_q) \cap \mathrm{Ber}_{r'} = \nu_r \cup \mathrm{Ber}_{q'}.$$

To finish this proof, we now only need to prove the claim.

[2]Note that we ask that $q_e < 1$ for every edge e

[3]The proof we present for this proposition was found in a discussion with Gemini Pro version 3.1.

Proof of Claim 3.4. As we can first condition on $\xi \sim \mu$, it is enough to prove this proposition for $\mu = \delta_\xi$ for every $\xi \in \{0,1\}^E$. Let us work in this case and define $\eta_s \sim \mathrm{Ber}_s$ for $s \in \{r, q, r', q'\}$ all independent. we distinguish two cases.

- If $e \in \xi$, note that $r' = 1 - (1-r)(1-q')$ and let us compute

$$\mathbb{P}(e \in (\xi \cap \eta_r) \cup \eta_{q'}) = 1 - (1-r_e)(1-q'_e) = r'_e,$$
$$\mathbb{P}(e \in (\xi \cup \eta_q) \cap \eta_{r'}) = r'_e.$$

(For the first equation, recall that $q'_e = r'_e q_e$.)

- If $e \notin \xi$, we have now

$$\mathbb{P}(e \in (\xi \cap \eta_r) \cup \eta_{q'}) = q'_e,$$
$$\mathbb{P}(e \in (\xi \cup \eta_q) \cap \eta_{r'}) = q_e r'_e = q'_e.$$

As the laws are equal for each $e \in E$ and they are independent, we conclude. □

□

We have just seen that the property of the dual weak domination is stable under taking intersection; not surprisingly, a similar result holds for the almost-equivalent-but-weaker condition that $\mu(\mathsf{A}_F) \le \nu(\mathsf{A}_F)$ for all $F \subset E$. Such a result will be useful in Section 4.4.4, where we show that this weaker property implies dual weak domination.

Proposition 3.5. *Let μ, ν be measures such that $\mu(\mathsf{A}_F) \ge \nu(\mathsf{A}_F)$ for all $F \subset E$. Then $\mu_p(\mathsf{A}_F) \ge \nu_p(\mathsf{A}_F)$ for all parameter p. Furthermore, if the inequality is strict for singletons, we have strict inequalities $\mu_p(\mathsf{A}_F) > \nu_p(\mathsf{A}_F)$ for any parameter $p < 1$ and non-empty subset F; in particular, $(\mu_p)^c$ weakly dominates $(\nu_p)^c$.*

Proof. Indeed, upon conditioning on $\omega \sim \mathrm{Ber}_p$ first, one sees that

$$\mu_p(\mathsf{A}_F) = \sum_{H \subset F} \mathrm{Ber}_p(F \cap \omega = H)\mu(\mathsf{A}_H). \tag{3.4}$$

An analogue equation holding for ν allows us to deduce the inequality $\mu_p(\mathsf{A}_F) \ge \nu_p(\mathsf{A}_F)$.

For the second part, assume that the inequality is strict for $\{e\} \subseteq F$. Then, since $p < 1$ the probability $\mathrm{Ber}_p(F \cap \omega = \{e\})$ is not be zero and therefore the equality cannot happen. □

3.2. **Weak forms of the FKG inequality.** We now turn to weak forms of the FKG inequality. The notions from the previous subsection admit natural analogues in the setting of a single measure satisfying a weak FKG condition, rather than two measures connected by stochastic domination.

3.2.1. *Weak FKG.* We start by noting that if μ satisfies the FKG inequality, then μ_p still has it (see Proposition 3.6), but the converse is not true. It makes natural the analogous definition:

Definition 4 (Weak FKG inequality)**.** *Take $p \in (0,1]^E$. We say that a probability measure μ satisfies the p–weak FKG inequality if μ_p satisfies the classical FKG inequality, that is to say, for all f, g increasing functions*

$$\mu_p(fg) \ge \mu_p(f)\mu_p(g).$$

Again, we say that μ satisfies the weak FKG inequality if there is some parameter $p \in (0,1]^E$ such that μ satisfies the p–weak FKG inequality.

In this context, we have the following analogue of Proposition 3.1 for p–stochastic domination.

Proposition 3.6. *Let μ measure on $\{0,1\}^E$, and $p \in [0,1]^E$ a parameter.*

(i) if μ satisfies the FKG inequality, then μ_p also has the FKG property.
(ii) if μ satisfies the weak FKG, then $\mu(\forall_{F \cup F'}) \ge \mu(\forall_F)\mu(\forall_{F'})$ for all $F, F' \subset E$.
(iii) if for any $F, F' \subset E$, $\mu(\forall_{F \cup F'}) \ge \mu(\forall_F)\mu(\forall_{F'})$, and the inequality is strict for any nonempty F, F', then μ has the weak FKG property.

The proof shares many similarities with the one of Proposition 3.1.

Proof. We start proving (i). First, note that the product measure $\mu \otimes \mathrm{Ber}_p$ satisfies the FKG inequality, as it is a product of two measures satisfying it. We conclude by using that the pushforward through an increasing map of a measure that has the FKG property also satisfies the FKG property.

To prove (ii), we restrict to disjoint F, F' as in the general case, replacing F' by $F' \backslash (F \cap F')$ and using the disjoint case yields a stronger statement. Now, for disjoint sets F, F', we have that for all $p \in (0,1]^E$

$$\mu(\forall_{F\cup F'}) \geq \mu(\forall_F)\mu(\forall_{F'}) \iff \mu_p(\forall_{F\cup F'}) \geq \mu_p(\forall_F)\mu_p(\forall_{F'}).$$

Thus if there is $p \in (0,1]^E$ such that μ_p satisfies FKG property, these inequalities are true for μ_p whence for μ.

For the third point, as in the proof of Proposition 3.1, it is enough to show that for any two non trivial increasing events A, B, it is true that $\mu_p(A \cap B) - \mu_p(A)\mu_p(B) > 0$ for some $p \in (0,1]^E$ constant. To this end, we make use of equation (3.2). Let k, ℓ be the minimal cardinal of a witness of A, B respectively, and let $\{F\}, \{F'\}, \{H\}$ be the sets of witnesses of cardinal k, ℓ, m respectively of $A, B, A \cap B$ respectively. Note that $m \leq k + \ell$.

Let us separate by cases. When $m < k + \ell$, we have that $\mu_p(A)\mu_p(B) \ll \mu_p(A \cap B)$ for $p \ll 1$. Thus the result is true in this case.

We are left with the case $m = k + \ell$. In this case call M_A, M_B and $M_{A\cap B}$ the set of with minimal cardinal of A, B and $A \cap B$ respectively. Note that the equality implies that for all $F \in M_A$ and $F' \in M_B$, we have that $F \cap F' = \emptyset$. As a consequence, the union

$$M_A \times M_B \xrightarrow{F,F' \mapsto F\cup F'} M_{A\cap B}$$

is an injection[4]. Indeed, if $F \cup F' = F' \cup F''$, since $F \cap F'' = \emptyset$ we have $F \subset (F' \cup F'') \backslash F'' = F'$ and by symmetry $F' \subset F$ so that $F = F'$; then $F' = F''$ follows. We can then write

$$\begin{aligned}
\mu_p(A \cap B) &= p^m \sum_{H \in M_{A\cap B}} \mu(\forall_H) + o(p^m) \\
&\geq p^{k+\ell} \sum_{\substack{F \in M_A \\ F' \in M_B}} \mu(\forall_F \cap \forall_{F'}) + o(p^m) \\
&> p^k \sum_{F \in M_A} \mu(\forall_F) \times p^\ell \sum_{F' \in M_B} \mu(\forall_{F'}) + o(p^m) = \mu_p(A)\mu_p(B) + o(p^m),
\end{aligned}$$

We conclude using the strict inequality and by taking $p > 0$ small enough. □

Remark 3.7. As explained in Remark 3.2 for weak domination, strict inequalities in the third item are needed to ensure weak FKG. Indeed, if $E = \{1,2,3\}$, and μ is the law of $\mathrm{Ber}_{1/2}$ conditioned to $\{\omega_1 + \omega_2 + \omega_3 \text{ odd}\}$, then one can check that μ satisfies the large inequalities in the third item, but for any parameter p, $\mu_p(A \cap \forall_{\{3\}}) < \mu_p(A)\mu_p(\forall_{\{3\}})$ for the increasing event $A = \{\omega_1 + \omega_2 > 0\}$.

3.2.2. *Weak*† *FKG inequality.* Now, as we did for the monotonicity, we introduce a type of FKG inequality which is weaker than the classical version but stronger than the weaker one. Similarly as in Subsection 3.1.2, it is convenient to introduce first the dual weak FKG property.

Definition 5 (Dual weak FKG)**.** *For a probability measure μ on $\{0,1\}^E$ and a parameter $q \in [0,1)^E$, we say that μ has the dual q–weak property if μ^c has the $(1-q)$–weak FKG property. If there exists such a parameter q, μ is said to have the dual weak FKG property.*

In the same way than for dual weak domination, this property can be rewritten in term of the measure μ: μ has dual q–weak FKG if $\mu \cup \mathrm{Ber}_q$ has FKG, or has dual weak FKG if there exists some $q < 1$ for which $\mu \cup \mathrm{Ber}_q$ has FKG.

By Proposition 3.6, this property implies $\mu(\mathsf{A}_F \cap \mathsf{A}_{F'}) \geq \mu(\mathsf{A}_F)\mu(\mathsf{A}_{F'})$ for any $F, F' \subset E$ and is implied by the strictness of these equalities for all non-empty subsets F, F'.

After this preliminary and the analogous definitions for the weak forms of dominations, the introduction of the weak† FKG property will not come as a surprise:

[4]Beware, it is *not* true that it is a bijection in the general case - however we don't need it

Definition 6 (Weak† FKG)**.** *For parameters $0<p,q\le 1$, we say that a measure μ satisfies the p,q–weak† FKG inequality if μ_p and $(\mu^c)_{1-q}$ both satisfy the FKG inequality. We say that μ has the weak† FKG property if there are parameters $p\in(0,1]^E, q\in[0,1)^E$ such that μ satisfies the p,q–weak† FKG inequality.*

Let us check that the weak † FKG property is stable under intersection with a Bernoulli measure.

Proposition 3.8 (Monotonicity of weak† FKG property)**.** *Assume that there are parameters $0<p,q\le 1$ such that μ satisfies the p,q–weak† FKG inequality. Then, for any parameter $0<r\le 1$, μ_r has the p',q'–weak† FKG property, for $p'=p/r\wedge 1$ and $q'=qr/(1-q(1-r))$.*
In particular, weak† FKG is stable under intersection with Ber_r*.*

Proof. The proof is very similar to the proof of Proposition 3.3, and relies as well on Claim 3.4. First, it is direct that μ_r has p'–weak FKG since this is equivalent to μ having $p'r$–weak FKG, which is true since $p'r\le p$. On the other hand, to show that μ_r^c has $(1-q')$–weak FKG, or equivalently $\mu_r\cup\mathrm{Ber}_{q'}$ has FKG, we can note that by Claim 3.4 this amounts to show that $(\mu\cup\mathrm{Ber}_q)\cap\mathrm{Ber}_{r'}$ has FKG, where $r'=r/(1-q(1-r))$. However, the FKG property is stable under taking intersection with the Bernoulli measure $\mathrm{Ber}_{r'}$ hence the result is true by assumption on μ. □

To conclude this section, we prove an additional monotonicity which will be key in Section 4 and more specifically the paragraph 4.4.4 to understand the critical parameter for weak†-FKG in the context of the energy field of Ising model. In order to introduce this inequality, recall that if μ has the weak FKG property, then for all $F,F'\subseteq E$ and all parameters $p\in[0,1]^E$, $\mu_p(\forall_{F\cup F'})\ge\mu_p(\forall_F)\mu_p(\forall'_F)$. We would also like to show that this is also true for the events of type A_F. This may look as a consequence of Proposition 3.8, however it does not follow from it.

Proposition 3.9. *Let μ be a measure such that $\mu(\mathsf{A}_F\cap\mathsf{A}_{F'})\ge\mu(\mathsf{A}_F)\mu(\mathsf{A}_{F'})$ for all $F,F'\subset E$. Then $\mu_p(\mathsf{A}_F\cap\mathsf{A}_{F'})\ge\mu_p(\mathsf{A}_F)\mu_p(\mathsf{A}_{F'})$ for all parameter p. Moreover, if the inequality is strict for singletons $\{e\},\{e'\}$, for any parameter $p<1$ we have strict inequalities $\mu_p(\mathsf{A}_F\cap\mathsf{A}_{F'})>\mu_p(\mathsf{A}_F)\mu_p(\mathsf{A}_{F'})$ for any non-empty F,F'; in particular, μ_p^c has weak FKG.*

Proof. As in the proof of the point (ii) of Proposition 3.6, we only need to focus on the case where F,F' are disjoint. Using the formula (3.4) for $F\sqcup F'$, we get

$$\mu_p(\mathsf{A}_{F\cup F'})=\sum_{H_1\subset F}\sum_{H_2\subset F'}\mathrm{Ber}_p\big((F\cup F')\cap\omega=H_1\cup H_2\big)\mu(\mathsf{A}_{H_1\cup H_2}).$$

Now, if use the assumption on μ, and formula (3.4) once again for F,F', we get precisely $\mu_p(\mathsf{A}_F\cap\mathsf{A}_{F'})\ge\mu_p(\mathsf{A}_F)\mu_p(\mathsf{A}_{F'})$.

In case that the inequality is strict for any singleton, since $p<1$ the probability $\mathrm{Ber}_p(F\cap\omega=\{e\})$ is not be zero and therefore the inequality is strict. □

4. On Ising's energy field

The objective of this section is to study weak versions of monotonicity and FKG for the energy field of the Ising model. For $G=(V,E)$ a finite graph with positive coupling constants $J=(J_e)_{e\in E}$, recall that the energy field of Ising on G at inverse temperature J is the random variable

$$\xi(\sigma)=\Big(\mathbb{1}(\sigma_x=\sigma_y)\Big)_{(xy)\in E},\tag{4.1}$$

where σ is distributed as an Ising model on G. We write the law of ξ as $\mathcal{E}_J=\mathcal{E}_J^G$. Alternatively, one can see that for any possible configuration ξ,

$$\mathcal{E}_J(\xi)\propto\prod_{e\,:\,\xi_e=0}e^{-2J_e}.\tag{4.2}$$

Here, by a possible configuration, we mean that there is σ such that $\xi=\xi(\sigma)$. Note that (4.2) implies that ξ has the law of a percolation configuration of parameter $p=(p_e)_{e\in E}$ with $p_e/(1-p_e)=e^{2J_e}$, conditioned on the event that the result happens to be the energy field (i.e. the gradient) of some vertex configuration σ.

It is known that the measures $\mathcal{E}_J$ are in general *not* increasing in (any of the component of) J in the sense of stochastic domination [Häg96] ; they also do not always satisfy the FKG inequality [RK22][5]. The objective of this section is to first show that weak forms of such properties, as defined in Section 3, are still true. Then, we find the exact thresholds for the parameters $p = p(J) \in [0,1]^E$ for which $(\mathcal{E}_J)_p = \mathcal{E}_J \cap \mathrm{Ber}_p$ is monotone or is weak† monotone. The same will be done for FKG and weak† FKG. Since we use a lot the measures $(\mathcal{E}_J)_p$ for distinct parameters J, p, it is convenient to write

$$\mathcal{E}_{J,p} := (\mathcal{E}_J)_p$$

to ease the writing.

4.1. **Results.** In this section, we plan to show two results regarding the threshold for the monotonicity properties discussed in Section 3. The first result states that the threshold for monotonicity and FKG is the FK-percolation parameter associated to J is $p(J) := 1 - e^{-2J}$.

Theorem 4.1 (Threshold for monotonicity and FKG property). *Let $G = (V, E)$ be a finite graph with coupling constants J. Then for any $p \leq p(J)$, $\mathcal{E}_J$ has the p–weak FKG property and for any $J_2 \geq J_1 \geq J$, $\mathcal{E}_{J_1} \preccurlyeq_p \mathcal{E}_{J_2}$.*

Furthermore, take $n \in \mathbb{N}$, $J \in (\mathbb{R}^+)^n$, and $p \in (0,1]^{n+1}$ such that

$$\sum_{k=1}^{n} \frac{p_k - p(J)_k}{p_k} > 1. \tag{4.3}$$

Then, there exists a graph G with $n+1$ edges with coupling constant J_G that satisfies the following:

- *If we numerate the edges of G, J_G restricted to its first n edges, is equal to J.*
- *Its energy field does not satisfy the p–weak FKG property*
- *There is $J'_G \geq J_G$ such that $\mathcal{E}_{J'_G}$ does not p–weakly dominates $\mathcal{E}_{J_G}$.*

Remark 4.2. Assumption (4.3) might seem complicated. In reality, it is not very restrictive and justifies the vague statement of Theorem 1.2 that one cannot take $p > p(J)$ such that in general $\mathcal{E}_{J,p}$ is either stochastically increasing in J or has the FKG property. Indeed, if one takes p, J to be constant on the edges, Inequality (4.3) is satisfied as soon as

$$\sharp E > \frac{1}{p - p(J)}.$$

In other words, if we take p close to $p(J)$ the "counter-example graph" G exists as soon as we allow ourselves enough edges. However, note that we are not ruling out the possibility that for any graph G there is a neighbourhood depending on G of parameters around $p(J)$ such that FKG property and weak stochastic domination in J still holds.

Remark 4.3 (A remark about weak monotonicity of the energy field.). Though it had not appeared previously in such a form, the fact that the energy field of the Ising model was weakly monotone in the coupling constants was already known: indeed, by Proposition 3.1, it is implied by monotonicity of the family of numbers $\left(\mathcal{E}_J(\forall_F)\right)_{F \subset E}$ in the coupling constants, a fact that can be seen to be a consequence of the classical monotonicity of the correlation functions of the spins, as explained below.

Proposition 4.4 (Folklore). *For $F \subset E$ non-empty, the probability $\mathcal{E}_J(\forall_F)$ is nondecreasing in (any component of) J.*

(Note that to apply Proposition 3.1 we need strict inequalities which are not true for some graphs G. However, one can easily reduce the analysis to graphs where strict monotonicity holds and then use it. See Subsection 4.4.4 for a similar treatment.)

Proof. Indeed, $\mathcal{E}_J(\forall_F) = \left\langle \prod_{(xy)\in F} \frac{1}{2}(1 + \sigma_x\sigma_y)\right\rangle_J = 1/2^{\#F} \sum_A \langle \sigma_A \rangle_J$ where A runs on some subsets of the vertices V, and σ_A is a shorthand for $\prod_{x\in A} \sigma_x$. However the last term is well known to be increasing in J. □

[5]in this reference it is stated for the loop $O(1)$ model which is the dual measure of the energy field

Unfortunately this reasoning does not allow one to give an explicit p for which p–weak monotonicity holds, thus preventing to have an analogous result for the infinite volume measures, a statement we can deduce from our Theorem.

Coming back to the proof, as explained in the Introduction, the $p(J)$–weak monotonicity derives from Theorem 1.1 which in turn is proven in Section 2. The FKG part is not new: it is well known that $\mathrm{FK}_{p(J),2} = \mathcal{E}_J \cap \mathrm{Ber}_{p(J)}$ has FKG. Thus, the main challenge of this theorem is to find the counterexample graph G, this is done in Section 4.2. The ideas there seem new to us.

Let us note that Theorem 4.1 can be read as a result on the threshold for FKG and stochastic monotonicity for the measures $\mathcal{E}_{J,p}$. We are also interested in the study of the threshold on the parameter p for the measures $\mathcal{E}_{J,p}$ but changing FKG and stochastic monotonicity by weak† FKG and weak† monotonicity. This new threshold must be at least as large than the previous threshold and the following theorem exhibits its exact value which is

$$p^\dagger = p^\dagger(J) := 1 - e^{-4J} = 1 - (1 - p(J))^2 > p(J).$$

Theorem 4.5 (Threshold for weak† domination and weak† FKG)**.** *Let $G = (V, E)$ a graph with coupling constants J. Then for every parameter $p < p^\dagger(J)$, the measure $\mathcal{E}_{J,p}$ satisfies the weak† FKG inequality and for any $J_2 \geq J_1 \geq J$, we have that $\mathcal{E}_{J_1,p} \preccurlyeq^\dagger \mathcal{E}_{J_2,p}$.*

Furthermore, there is a graph G such that for all $p \geq p^\dagger(J)$, we have that $\mathcal{E}_{J,p}$ does not satisfy the weak† FKG inequality and there is $J' \geq J$ such that $\mathcal{E}_{J',p}$ does not weakly† dominate $\mathcal{E}_{J,p}$.

The "counter-example" graph can actually be chosen to be very simple: one may take C_4 the cycle graph with four vertices and edges. Note that if we changed weak† by weak in Theorem 4.5, the threshold parameter is trivially 1 as this follows from Theorem 4.1. The fact that this is not the case for the weak† version means in particular that the energy field does not dually weakly increases, neither does it have dual weak FKG.

Since weak† domination/ FKG is really having both weak and dual weak domination/ FKG, and that we already know the "primal" part by Theorem 4.1, we need only to focus on the dual notions.

Let us now comment on the percolation $\mathcal{E}_J^\dagger = \mathcal{E}_{J,p^\dagger(J)}$ at the threshold point $p^\dagger(J)$: the fact that at $p = p^\dagger$ the weak† properties do not hold is not a contradiction: indeed, weak† domination and weak† FKG are not closed under weak convergence of measures. However, as a corollary of Theorem 4.5 one must have

$$\mathcal{E}_{J,p^\dagger}(\mathsf{A}_F \cap \mathsf{A}'_F) \geq \mathcal{E}_{J,p^\dagger}(\mathsf{A}_F)\mathcal{E}_{J,p^\dagger}(\mathsf{A}'_F) \qquad \forall F, F' \subseteq E, \tag{4.4}$$

$$\mathcal{E}_{J,p^\dagger}(\mathsf{A}_F) \geq \mathcal{E}_{J',p^\dagger}(\mathsf{A}_F) \qquad \forall J' \geq J \tag{4.5}$$

because these properties are implied respectively by dual weak FKG and dual weak monotonicity, and are themselves close by weak convergence of measures. To prove Theorem 4.5, we actually first prove the inequalities above (4.4) and (4.5). This is the hardest part of the proof and it is the content of Section 4.4.3. Then, to conclude the proof, we show in Section 4.4.4 that these inequalities imply weak† FKG and weak† monotonicity for parameters p that are strictly smaller values of p.

Notations In the rest of the section, we use the notation $q = (q_e)_{e\in E} = e^{-2J}$, so that $p(J) = 1-q$ and $p^\dagger(J) = 1-q^2$. We write as well $q_F = \prod_{e\in F} q_e$ for $F \subset E$ a subset of edges, and $Z = Z_J$ for the partition function of the energy field defined by equation (4.2), i.e.

$$Z = Z_J = \sum_{F\in\mathcal{F}} q_F \tag{4.6}$$

where $\mathcal{F}$ is the subset of $\mathcal{P}(E)$ containing F if and only if there is a spin configuration $\sigma \in \{\pm\}^V$ such that F are the *closed* edges of the percolation $\xi(\sigma)$. Beware that q is a *decreasing* function of J.

4.2. **Upper bound for the weak threshold and end of the proof of Theorem 4.1.** In this section, we prove that $p(J) = 1 - e^{-2J}$ is indeed the threshold for weak FKG and weak monotonicity[6], concluding the proof of Theorem 4.1. Though the precise statement in Theorem 4.1 is a bit complicated, the "counterexample graph" for this situation is quite simple as we will choose the cycle graph on $n+1$ vertices again.

We start by proving a version of Russo's formula that is somehow an extension of Theorem 2.46 in [Gri06]. It allows us to deal at once with the monotonicity part and the FKG part.

Lemma 4.6. *Let A be an event that does not depend on edge e, a parameter $p \in (0,1]^E$ and $\omega \sim \mathcal{E}_{J,p}$. Then,*

$$\partial_{J_e}[\mathcal{E}_{J,p}(A)] = \frac{2}{p_e}\mathrm{Cov}_{J,p}(\mathbf{1}_A, \omega_e), \tag{4.7}$$

where covariance is taken with respect to the measure $\mathcal{E}_{J,p}$.

The proof of this formula follows the classical ideas already present in [Gri06].

Proof. Let us first compute $\partial_{J_e}\mathcal{E}_J(A)$. Note that by summing over possible configuration $\xi \subseteq E$

$$\begin{aligned}\partial_{J_e}\mathcal{E}_J(A) &= \frac{1}{Z_J}\partial_{J_e}\sum_{\xi\in A} e^{2\sum_{e\in E}\xi_e J_e} - \frac{1}{Z_J}\partial_{J_e}\sum_{\xi\in A} e^{2\sum_{e\in E}\xi_e J_e}\frac{\partial_{J_e}Z_J}{Z_J} \\ &= 2\mathcal{E}_J(\xi_e\mathbf{1}_A) - 2\mathcal{E}_J(A)\mathcal{E}_J(\xi_e) \\ &= 2\mathrm{Cov}_{J,1}(\mathbf{1}_A, \xi_e).\end{aligned}$$

Now, for $F \subseteq E$ we denote the event

$$A^F := \{\xi \subset E : \xi \cap F \in A\}.$$

Take a percolation $\eta \sim \mathrm{Ber}_p$ independent of $\xi \sim \mathcal{E}_J$ and note that $\mathcal{E}_{J,p}(A) = \mathrm{Ber}_p[\mathcal{E}_J(A^\eta)]$. This implies that

$$\partial_{J_e}[\mathcal{E}_{J,p}(A)] = \mathrm{Ber}_p\left[\partial_{J_e}\mathcal{E}_J(A^\eta)\right] = 2\mathrm{Ber}_p\left[\mathcal{E}_J(\xi_e\mathbf{1}_{A^\eta})\right] - 2\mathcal{E}_J(\xi_e)\mathrm{Ber}_p[\mathcal{E}_J(A^\eta)].$$

Now, since A does not depend on e, the random variables η_e and $\xi_e\mathbf{1}_{A^\eta}$ are independent. Whence,

$$p_e\mathrm{Ber}_p\left[\mathcal{E}_J(\xi_e\mathbf{1}_{A^\eta})\right] = \mathrm{Ber}_p[\mathcal{E}_J(\eta_e\xi_e\mathbf{1}_{A^\eta})] = \mathcal{E}_{J,p}[\omega_e\mathbf{1}_A].$$

We conclude from the fact that $p_e\mathcal{E}_J(\xi_e) = \mathcal{E}_{J,p}(\omega_e)$ and $\mathcal{E}_J(\xi_e)\mathrm{Ber}_p[\mathcal{E}_J(A^\eta)] = \mathcal{E}_{J,p}(A)$. ☐

Consider the cycle graph $G = (V,E) = C_{n+1}$ where edge set is identified $E = \{0,\dots,n\}$ and fix $J \in (\mathbb{R}_{>0})^E$, and $p \in (0,1]^E$ such that they satisfy assumption (4.3). Recall that we use the notation $q = 1 - p(J)$, so that it writes

$$\sum_{e=1}^{n}\frac{q_e + p_e - 1}{p_e} > 1.$$

The numbers J_0, p_0 are arbitrary. We wish to prove that $\mathcal{E}_{J,p}$ doesn't have the FKG property nor is stochastically increasing in any of the J_e coordinates.

We start by establishing a decomposition of $\mathcal{E}_J$ as a convex combination of two Bernoulli measures, remarkably one of these measures has a parameter strictly bigger than one which implies that it is *not a probability measure*. More precisely,

Lemma 4.7. *Take $C_1 = \prod_{e=0}^n(1+q_e)$, $C_{-1} = \prod_{e=0}^n(1-q_e)$ and $Z = \sum_{F\in\mathcal{F}} q_F$ where $\mathcal{F} \subset \mathcal{P}(E)$ is the set of subsets of E of even cardinal and $q_F = \prod_{e\in F} q_e$ for $F \subset E$. We have that $C_1 + C_{-1} = 2Z$ and*

$$\mathcal{E}_J = \frac{1}{2Z}\left(C_1\cdot\mathrm{Ber}_{\frac{1}{1+q}} + C_{-1}\cdot\mathrm{Ber}_{\frac{1}{1-q}}\right) \tag{4.8}$$

as measures.

[6]The proof we present for this proposition was found in a discussion with Gemini Pro version 3.1.

As we stated before the proof we are abusing notation as $1/(1-q) > 1$. In this context, for $r \in \mathbb{R}^E$, we define Ber_r to be the (signed) measure on $\{0,1\}^E$ where

$$\text{Ber}_r(\eta) = \prod_e r_e^{\eta_e}(1-r_e)^{1-\eta_e}.$$

It is straightforward to see that Ber_r is a probability measure if and only if $r \in [0,1]^E$ and that $\text{Ber}_r(\{0,1\}^E) = 1$ for all $r \in \mathbb{R}^E$.

Proof. In the case of the cycle graph, $\xi \in \{0,1\}^E$ is the gradient of a vertex configuration if and only if $\xi^c = 1-\xi$ is such that $\sum_e \xi_e^c$ is even. In this context, we see that $\mathcal{E}_J$ can be defined by conditioning $\xi \sim \text{Ber}_{\frac{1}{1+q}}$ on the event that $\{\xi^c \in \mathcal{F}\}$. It also makes sense to define the measure $\mathcal{E}'_J$, to be the the law of $\xi \sim \text{Ber}_{\frac{1}{1+q}}$ conditioned on the event $\{\xi^c \notin \mathcal{F}\}$. Note that in these cases, for each of these cases the partition function are up to a multiplication by C_1^{-1}

$$Z = \sum_{F \in \mathcal{F}} q_F \qquad \text{and} \qquad Z' := \sum_{F \notin \mathcal{F}} q_F,$$

respectively.

Let us see that we can decompose the Bernoulli measures by using $\mathcal{E}_J$ and $\mathcal{E}'_J$ as follows:

$$\text{Ber}_{\frac{1}{1+q}} = \frac{1}{C_1} \cdot \big(Z \cdot \mathcal{E}_J + Z' \cdot \mathcal{E}'_J\big) \quad \text{and} \quad \text{Ber}_{\frac{1}{1-q}} = \frac{1}{C_{-1}} \cdot \big(Z \cdot \mathcal{E}_J - Z' \cdot \mathcal{E}'_J\big). \tag{4.9}$$

Indeed, the first equality is just the formula of total probability. The second one could also be interpreted as a formula of total probability, but as it is not a probability measure we need to prove it by hand. Note that for any configuration $\xi \in \{0,1\}^E$, $Z\mathcal{E}_J(\xi) - Z'\mathcal{E}_J(\xi)$ is equal to

$$\prod_e q_e^{1-\xi_e}\mathbf{1}_{\xi^c \in \mathcal{F}} - \prod_e q_e^{1-\xi_e}\mathbf{1}_{\xi^c \notin \mathcal{F}} = (-1)^{\xi^c \notin \mathcal{F}} \prod_e q_e^{1-\xi_e}.$$

By summing over all configurations $\xi \in \{0,1\}^E$, we see that $Z + Z' = C_1$ and $Z - Z' = C_{-1}$; thus $C_1 + C_{-1} = 2Z$. Furthermore, by multiply (4.9) by C_1, C_{-1} respectively and summing them, we obtain (4.8). □

Before proving the second part of Theorem 4.1, let us make some comments regarding signed measures with mass 1, as this is not a common object in Probability theory but many well-known formulae are equally true in this context. In particular, if we have a signed measure ν with total mass equal to 1, and $f, g \in L^2(|\nu|)$ we can define

$$\text{Cov}_\nu(f,g) := \int fg d\nu - \int f d\nu \int g d\nu.$$

Then, we can check that classical formulae for the covariance still hold in this setup. For example

i) If ν is a product measure and f and g depend on different coordinates, $\text{Cov}_\nu(f,g) = 0$.
ii) If $\nu = \gamma\nu_1 + (1-\gamma)\nu_2$, where $\gamma \in \mathbb{R}$ and ν_1, ν_2 are signed measures with mass equal to 1

$$\text{Cov}_\nu(f,g) = \gamma\text{Cov}_{\nu_1}(f,g) + (1-\gamma)\text{Cov}_{\nu_2}(f,g) + \gamma(1-\gamma) \cdot \big(\nu_2(f) - \nu_1(f)\big) \cdot \big(\nu_2(g) - \nu_1(g)\big). \tag{4.10}$$

This is the extension of the law of total covariance.

We now have the tools to show the second part of Theorem 4.1. The strategy is to construct an increasing event A independent of edge 0 such that $\text{Cov}(\mathbf{1}_A, \omega_0) < 0$. This prevents $\mathcal{E}_J$ from having the p–weak FKG property, and, thanks to Lemma 4.6, $\mathcal{E}_J$ is not p–weak monotone in J, precisely the two statements we are aiming. A key technical tool that we use in the proof concerns the intersections[7] or Bernoulli measures: $\text{Ber}_r \cap \text{Ber}_{\tilde r} = \text{Ber}_{r\tilde r}$ for any parameters $r, \tilde r \in \mathbb{R}^E$.

[7]Recall the intersection of measures defined at the beginning of Section 2.

Proof of Theorem 4.1 second paragraph. Define the event $A := \{\sum_{e=1}^n \xi_e \geq n-1\}$. Using the notations of Lemma 4.7, we define $\alpha = C_1/(2Z)$ and $\beta = C_{-1}/(2Z)$ and note that $\alpha+\beta = 1$. Now, we use Lemma 4.7 and the fact that intersection of two Bernoulli measures is a Bernoulli measure to see that

$$\mathcal{E}_{J,p} = \alpha \cdot \mathrm{Ber}_{\frac{p}{1+q}} + \beta \cdot \mathrm{Ber}_{\frac{p}{1-q}}. \tag{4.11}$$

As $\beta = 1-\alpha$, we can use (4.10) with $\nu_1 = \mathrm{Ber}_{p/(1+q)}$ and $\nu_2 = \mathrm{Ber}_{p/(1-q)}$, $f = \mathbf{1}_A$ and $g = \xi_0$, to see that

$$\mathrm{Cov}_{J,p}(\mathbf{1}_A, \xi_0) = \alpha\beta \frac{2p_0q_0}{(1+q_0)(1-q_0)}(\nu_2(A) - \nu_1(A)).$$

This is because $\mathrm{Cov}_{\nu_1}(f,g) = \mathrm{Cov}_{\nu_2}(f,g) = 0$ since ν_1, ν_2 are product measures, and $\nu_2(g) = p_0/(1-q_0)$ and $\nu_1(g) = p_0/(1+q_0)$. We now prove that $\nu_2(A) < 0$, since the constant multiplying $(\nu_2(A) - \nu_1(A))$ is strictly positive, this is enough to show that $\mathrm{Cov}_{J,p}(\mathbf{1}_A, \xi_0) < 0$.

To show the negativity of $\nu_2(A) = \mathrm{Ber}_{p/(1-q)}(A)$, let use the formula for Bernoulli measures: if $r = p/(1-q)$,

$$\mathrm{Ber}_r(A) = \sum_{e=1}^n \mathrm{Ber}_r\big(A, \omega_e = 0\big) + \mathrm{Ber}_r\left(\Big\{\sum_{e=1}^n \omega_e = n\Big\}\right) = \prod_{e=1}^n r_e \cdot \left[\sum_{e=1}^n \frac{1-r_e}{r_e} + 1\right]. \tag{4.12}$$

As

$$\frac{1-r_e}{r_e} = -\frac{p_e + q_e - 1}{p_e}$$

whose sum over $e \in \{1,..,n\}$ is < -1, we obtain that $\nu_2(A) < 0$ and thus $\mathrm{Cov}_{J,p}(\mathbf{1}_A, \xi_0) < 0$. With this we found a counterexample for the FKG inequality and by Lemma 4.6 we see that $\mathcal{E}_{J,p}(A)$ is not increasing in J_0. □

Remark 4.8. Lemma 4.7 decomposing the energy field into a sum of Bernoulli measures can be generalized to any finite graph (not only the cycle) and seems fruitful to study. However, since we do not need it further than for the special case of the cycle, we do not present it.

It is clear with the decomposition used in the proof that $p(J) = 1-q$ is a threshold: it is the largest p for which the decomposition (4.11) becomes a convex combination of *probability distributions*. This property holds true for any finite graph.

4.3. **No weak† domination and FKG at and above the threshold $p^\dagger$.** In this section, we prove that $p^\dagger(J) = 1 - e^{-4J}$ is an upper bound for the threshold of the weak† FKG and weak† monotonicity. Since, by Propositions 3.3 and 3.9, it is enough to deal at the level $p = p^\dagger(J)$[8].

Let $G = C_4$, with edge set $E = \{0,1,2,3\}$, and let $q_e = e^{-2J_e} \in (0,1)$. We write

$$Z = Z_{even} := \sum_{F\subset E:\ |F|\ even} q_F, \qquad Z_{odd} := \sum_{F\subset E:\ |F|\ odd} q_F.$$

Take $\mu_J = \mathcal{E}_J^\dagger = \mathcal{E}_{J,p^\dagger(J)}$ and fix an arbitrary parameter $r \in [0,1)^E$. Set $s_e = 1 - r_e > 0$ and

$$\mu_J^r := \mu_J \cup \mathrm{Ber}_r.$$

Since an edge is closed for μ_J^r if and only if it is closed for μ_J and is not opened by the independent Bernoulli field, for every $F \subset E$,

$$\mu_J^r(\mathsf{A}_F) = s_F \mu_J(\mathsf{A}_F). \tag{4.13}$$

Moreover, on C_4, the admissible closed sets for the energy field are exactly the even subsets of E. Hence, directly from the definition of $\mathcal{E}_J^\dagger$,

$$\mu_J(\mathsf{A}_F) = \frac{q_F}{Z} \sum_{H\subset E:\ |H|\ even} q_{H\Delta F} = \begin{cases} q_F, & |F|\ even, \\ q_F \dfrac{Z_{odd}}{Z}, & |F|\ odd. \end{cases} \tag{4.14}$$

We now consider the decreasing events

$$A := \mathsf{A}_{\{1,3\}} \cup \mathsf{A}_{\{2,3\}}, \qquad B := \mathsf{A}_{\{0\}}.$$

[8]This counter-example was found during the course of a discussion with ChatGPT 5.5.

Using (4.13) and (4.14),

$$\mu_J^r(A) = s_3q_3(s_1q_1 + s_2q_2) - s_1s_2s_3q_1q_2q_3\frac{Z_{odd}}{Z},$$
$$\mu_J^r(B) = s_0q_0\frac{Z_{odd}}{Z},$$
$$\mu_J^r(A\cap B) = s_0s_3q_0q_3(s_1q_1 + s_2q_2)\frac{Z_{odd}}{Z} - s_0s_1s_2s_3q_0q_1q_2q_3.$$

Therefore, using

$$Z^2 - Z_{odd}^2 = (Z + Z_{odd})(Z - Z_{odd}) = \prod_{e=0}^{3}(1-q_e^2),$$

we obtain

$$\mathrm{Cov}_{\mu_J^r}(\mathbf{1}_A, \mathbf{1}_B) = -\frac{s_0s_1s_2s_3q_0q_1q_2q_3}{Z^2}\prod_{e=0}^{3}(1-q_e^2) < 0. \tag{4.15}$$

Since A and B are decreasing events, the FKG inequality for μ_J^r would imply that this covariance is non-negative. Thus, for no parameter $r < 1$ does $\mathcal{E}_J^\dagger \cup \mathrm{Ber}_r$ satisfy FKG. In particular, $\mathcal{E}_J^\dagger$ does not have weak† FKG.

The same example rules out weak† monotonicity at the threshold. Indeed, the event A is decreasing, and its probability under μ_J^r satisfies

$$\partial_{J_0}\mu_J^r(A) = \frac{2s_1s_2s_3q_0q_1q_2q_3}{Z^2}\prod_{e=1}^{3}(1-q_e^2) > 0. \tag{4.16}$$

To see this, write $Z = A_0 + q_0B_0$ and $Z_{odd} = B_0 + q_0A_0$, where A_0 and B_0 are respectively the even and odd sums in the variables q_1, q_2, q_3; then

$$\partial_{J_0}\left(\frac{Z_{odd}}{Z}\right) = -\frac{2q_0}{Z^2}(A_0^2 - B_0^2) = -\frac{2q_0}{Z^2}\prod_{e=1}^{3}(1-q_e^2),$$

and (4.16) follows from the formula for $\mu_J^r(A)$ above. If the family $J \mapsto \mathcal{E}_J^\dagger \cup \mathrm{Ber}_r$ were stochastically increasing, the probability of every decreasing event would be non-increasing in each coordinate J_e. Equation (4.16) gives the opposite behaviour. Thus no $r < 1$ makes $\mathcal{E}_J^\dagger \cup \mathrm{Ber}_r$ increasing in J, and consequently the threshold value $p^\dagger$ cannot be included in the weak† domination statement.

Remark 4.9. As we just proved, the obstruction already appears on the cycle C_4. Actually, if the parameter p had been taken strictly above $p^\dagger$, one would have had the negative results even for the triangle C_3. Indeed, one can check that the inequalities (4.4),(4.5) would not hold for values of p strictly larger than $p^\dagger$.

4.4. **Lower bound for the weak† threshold.** We are now in position to deal with the first paragraph of Theorem 4.5, which is its harder part. Let us recall the notation $\mathcal{E}_J^\dagger := \mathcal{E}_{J,p^\dagger(J)}$. This measure plays a key role in the proof, a fact which should not come as a surprise since the proof of Theorem 4.1 strongly relies on the properties of $\mathcal{E}_{J,p(J)} = \mathrm{FK}_{p,2}$.

When $G = (V, E)$ is a planar graph, the proof of this result relies on the high temperature expansion of the Ising model on the dual graph. However for general graphs, we require a broader framework. We address this in Section 4.4.1, by introducing the Ising model on an abstract graph. While formalising the full Ising model is not strictly necessary, as the high temperature expansion alone would suffice, doing so provides a much clearer conceptual picture. As the following section is quite abstract, so the reader may skip it in a first reading and assume that the graph is planar.

Remark 4.10. The fact that we need to generalize the Ising model is reminiscent of the fact that the key coupling in [ALHL26] (Theorem 2.7.) seems at first glance to be related to planarity. Indeed, though they achieve a construction of their percolation model in the general case (in Definition 2.6.), it can also be defined in the special case of a planar graph as the dual of the double random current on the dual graph and some of their proofs become easier in this context. Our contructions could allow one to actually make sense of this for any graphs.

Note furthermore that in their equation (2.8), if one only takes the intersection $\xi(\sigma)\cap\eta$ (without the extra $\xi(\tilde{\sigma})$), one recovers our model $\mathcal{E}^{\dagger}$.

4.4.1. *Introduction and basic properties of the Ising model on an abstract graph.* In this paragraph, we introduce a general framework to define an Ising model, the reason for it will become clear later during the proofs.

Definition 7. *Let E be a finite set and $\mathcal{F}\subseteq\mathcal{P}(E)$. We say that $\mathcal{F}$ a* high-temperature structure *(HT structure) on E, if $(E,\mathcal{F})$ is a subgroup of $(\mathcal{P}(E),\Delta)$.*

Of course taking $\mathcal{F}$ to be either $\{\emptyset\}$ or $\mathcal{P}(E)$ gives us an HT example, but we think of them as begin trivial. For a graph $G=(V,E)$ there are two important HT structures. The first one is the *natural HT structure* which corresponds to

$$\mathcal{F}_G=\{\xi\subset E:\xi \text{ even}^{9} \text{ subgraph of } G\},$$

and can be thought of as the trace of all possible loops on G. The second one is the *dual HT structure*, and is based in (4.1)

$$\mathcal{F}_G^{*}:=\{\xi\subset E:\text{there is }\sigma:V\to\{\pm1\}\text{ s.t. }\xi^{c}=\xi(\sigma)\}. \tag{4.17}$$

(Note that ξ is the *complementary* of the energy field of an Ising configuration.) Indeed, $\mathcal{F}_G^{*}$ is an HT structure as $\sigma\mapsto\xi(\sigma)^{c}$ is a group homomorphism. Furthermore, if G is planar the natural HT structure of its dual G^{*} can be identified with $\mathcal{F}_G^{*}$.

As we stated at the beginning of this section, we want to define an abstract Ising model with edge sets on E. To do that we need to define the "values" of the model. This is done through the following definition.

Definition 8 (Space of spin configurations)**.** *Let E be a finite graph and $\mathcal{F}$ be an HT structure. We define the space of spin configurations as*

$$\mathcal{V}=\mathcal{V}_{\mathcal{F}}=\mathrm{Hom}(\mathcal{P}(E)/\mathcal{F},\{\pm1\}), \tag{4.18}$$

where $\{\pm1\}$ is seen as the group with two elements, and homomorphisms are homomorphisms of groups.

In other words, $\mathcal{V}$ can be canonically identified with the set of maps $\sigma:\mathcal{P}(E)\to\{\pm1\}$ such that $\sigma(F\Delta F')=\sigma(F)\sigma(F')$ for any $F,F'\subset E$, and $\sigma(\eta)=1$ if $\eta\in\mathcal{F}$. In fancy words, the set $\mathcal{V}$, seen as a group, is the Pontryagin dual of the abelian group $\mathcal{P}(E)/\mathcal{F}$.

The guiding intuition for this definition comes from taking a connected graph G and looking at $\mathcal{F}_G$ its natural HT map. In this case, one can naturaly identify $\mathcal{V}$ to the set $\{\pm1\}^{V}/\{\pm\}$ of the spin configurations on G up to a global flip. Indeed, to $\sigma\in\{\pm1\}^{V}$ one can associate the homomorphism $h(\sigma)$ defined by

$$\omega\subseteq E\mapsto[h(\sigma)](\omega)=\prod_{e=(xy)\in\omega}\sigma_x\sigma_y.$$

Note that if $\omega\in\mathcal{F}_G$, then $[h(\sigma)](\omega)=1$, thus $h(\sigma)$ can be thought of as an homomorphism from $\mathcal{P}(E)/\mathcal{F}_G$ to $\{\pm1\}$. Furthermore, as $h(\sigma)$ only depends on σ up to a global flip, h can be defined as a map from $\{\pm1\}^{V}/\{\pm\}$ to $\mathrm{Hom}(\mathcal{P}(E)/\mathcal{F},\{\pm1\})$. In the case of a connected graph, one can check that h is, in fact a bijection, as one can recover σ by fixing a spanning tree with a root and multiplying through the edges.

We can now define an Ising model on E equipped with a HT structure $\mathcal{F}$.

Definition 9 (Abstract Ising model)**.** *Let E be a finite set, $\mathcal{F}$ be an HT structure on E and positive coupling constants $J=(J_e)_{e\in E}\subseteq(\mathbb{R}^{+})^{E}$. An abstract Ising model on $(E,\mathcal{F})$ with coupling constant J, is the probability distribution on $\mathcal{V}_{\mathcal{F}}$ defined by*

$$\mathrm{Is}(\sigma)\propto e^{-H(\sigma)}, \tag{4.19}$$

where the Hamiltonian is defined as

$$H(\sigma)=-\sum_{e\in E}J_e\sigma(e).$$

Note that if we fix a graph $G = (V, E)$ and we define an abstract Ising model on the natural HT structure $\mathcal{F}_G$, then it is precisely the (free) classical Ising model up to a global flip through the identification h^{-1}. Furthermore for $A \subseteq V$, the classical correlation functions $\langle \sigma_A \rangle_{G,J}$ are interpreted naturally as $\langle \sigma(F) \rangle$ for $F \subset E$ such that $\partial F = A$.

Let us now write a high temperature expansion for the abstract Ising model. This is a generalisation of the classical case, which one can properly learn in, e.g. [FV17] (Section 3.7.3.)

Proposition 4.11 (High temperature expansion)**.** *Let σ be an abstract Ising model on $(E, \mathcal{F})$ with coupling constant J. Then for every $F \subseteq E$*

$$\big\langle \sigma(F) \big\rangle_J = \frac{\sum_{\eta \in \mathcal{F}} w(\eta \Delta F)}{\sum_{\eta \in \mathcal{F}} w(\eta)}, \tag{4.20}$$

where for $F' \subseteq E$

$$w(F') = \prod_{e \in F'} \tanh(J_e).$$

Proof. Let us compute $Z_J[F]$, the unnormalised probability of F. We define the constant $C = \prod_{e \in E} \cosh(J_e)$ and compute

$$\begin{aligned}
\sum_{\sigma \in \mathcal{V}} \sigma(F) e^{-H(\sigma)} &= \sum_{\sigma \in \mathcal{V}} \sigma(F) \prod_{e \in E} e^{J_e \sigma(e)} \\
&= C \sum_{\sigma \in \mathcal{V}} \sigma(F) \prod_{e \in E} \big(1 + \sigma(e) \tanh(J_e)\big) \\
&= C \sum_{\eta \subset E} w(\eta) \sum_{\sigma \in \mathcal{V}} \sigma(F \Delta \eta) \\
&= C \,\#\mathcal{V} \sum_{\eta \in \mathcal{F} \Delta F} w(\eta) = C \#\mathcal{V} \sum_{\eta \in \mathcal{F}} w(\eta \Delta F).
\end{aligned}$$

Here, in the third to last line we used that $\sum_{\sigma \in \mathcal{V}} \sigma(\eta') = \#\mathcal{V} \cdot \mathbf{1}_{\{\eta' \in \mathcal{F}\}}$. This is clear for $\eta' \in \mathcal{F}$ since $\sigma(\eta') = 1$ for any $\sigma \in \mathcal{V}$ by definition. If $\eta' \notin \mathcal{F}$ there is $\sigma_0 \in \mathcal{V}$ such that $\sigma_0(\eta') = -1$, as $\sigma \mapsto \sigma_0 \sigma$ is a bijection from $\mathcal{V}$ to itself, we see that $\sum_{\sigma \in \mathcal{V}} \sigma(\eta') = \sum_{\sigma \in \mathcal{V}} \sigma(\eta') \sigma_0(\eta') = -\sum_{\sigma \in \mathcal{V}} \sigma(\eta')$. Whence, we conclude that the sum is zero.

□

A direct corollary of the high temperature expansion of the abstract Ising model are the GKS inequalities.

Proposition 4.12 (GKS inequalities for the abstract Ising model)**.** *Let σ be an abstract Ising model on $(E, \mathcal{F})$. Then, for any $F, G \subset E$, one has $\langle \sigma(F) \rangle \geq 0$ and $\langle \sigma(F) \sigma(G) \rangle \geq \langle \sigma(F) \rangle \cdot \langle \sigma(G) \rangle$.*

Proof. The first inequality follows from (4.20); the second one can be proved exactly in the same way than the version for the classical Ising model (see e.g. Theorem 3.49. in [FV17] and the proof following it). □

Proposition 4.13 (Monotonicity in the HT structure)**.** *Let $\mathcal{F}_1, \mathcal{F}_2$ be two HT structures on E, with $\mathcal{F}_1 \subset \mathcal{F}_2$. Then for any coupling constants J and subset of edges F, one has $\langle \sigma(F) \rangle_J^{\mathcal{F}_1} \leq \langle \sigma(F) \rangle_J^{\mathcal{F}_2}$.*

Remark 4.14. To fix ideas, take a graph $G_1 = (V_1, E)$ and define $G_2 = (V_2, E)$ to be constructed by identifying V_1 along a subset of edges F'. In this case the natural HT structures can be compared as $\mathcal{F}_{G_1} \subseteq \mathcal{F}_{G_2}$ and thus our inequality corresponds to the fact that contracting edges increases correlation functions. The fact that $\mathcal{F}^*_{G_2} \subset \mathcal{F}^*_{G_1}$ and the subsequent inequality amounts this time to the fact that on the abstract dual one has "morally" less edges by contracting edges on the primal graph.

Proof. Let us start by using GKS inequality to see that for the HT structure $\mathcal{F}_1$ and any subset of edges H, F one has that

$$\sum_{\eta \in \mathcal{F}_1} w(\eta \Delta F) \sum_{\eta \in \mathcal{F}_1} w(\eta \Delta H) \leq \sum_{\eta \in \mathcal{F}_1} w(\eta) \sum_{\eta \in \mathcal{F}_1} w(\eta \Delta H \Delta F). \tag{4.21}$$

Take $\mathcal{H} = \{H_1, \ldots, H_k\} \subset \mathcal{P}(E)$ a complete set of representatives of the group $\mathcal{F}_2/\mathcal{F}_1$, and sum (4.21) over all $H \in \mathcal{H}$ to obtain

$$\sum_{\eta\in\mathcal{F}_1} w(\eta\Delta F) \sum_{H\in\mathcal{H}} \left(\sum_{\eta\in\mathcal{F}_1} w(\eta\Delta H) \right) \leq \sum_{\eta\in\mathcal{F}_1} w(\eta) \sum_{H\in\mathcal{H}} \left(\sum_{\eta\in\mathcal{F}_1} w(\eta\Delta H\Delta F) \right).$$

We conclude by using (4.20) and noting that

$$\langle\sigma(F)\rangle_J^{\mathcal{F}_2} = \frac{\sum_{H\in\mathcal{H}} \sum_{\eta\in\mathcal{F}_1} w(\eta\Delta H\Delta F)}{\sum_{H\in\mathcal{H}} \sum_{\eta\in\mathcal{F}_1} w(\eta\Delta H)}.$$

□

4.4.2. *The model $\mathcal{E}^\dagger$ and the abstract Ising model on the dual structure.* In this section, we show that the percolation model $\mathcal{E}_J^\dagger = \mathcal{E}_{J,p^\dagger}$ is related to the abstract Ising model generated by the dual HT structure of G. Of course, in the case of a planar graph, one can think of $\mathcal{E}_J^\dagger$ as a subset of the dual graph.

Let $F \subset E$, we show that $\mathcal{E}_J^\dagger(\mathsf{A}_F)$ is given by a particularly nice formula. Because of what follows, it is better to consider more generally the measure $\mathcal{E}_{J,p}$ and to specialize to $p = 1 - q^2 = 1 - e^{-4J}$ later on. We recall that for a subset $F \subseteq E$ and a function $s : E \to \mathbb{R}$, we write $s_F = \prod_{e\in F} s_e$.

Proposition 4.15. *Let $G = (V, E)$ a finite graph with coupling constants $J = (J_e)_{e\in E}$, $F \subset E$ a subset of the edge, and $p \in [0, 1]^E$ a parameter. Denoting $q = e^{-2J}$ and $r = 1 - p$, we have that*

$$\mathcal{E}_{J,p}(\mathsf{A}_F) = \frac{\sum_{\eta\in\mathcal{F}_G^*} q_\eta \times r_{F\setminus\eta}}{\sum_{\eta\in\mathcal{F}_G^*} q_\eta}. \tag{4.22}$$

Furthermore, when $p = p^\dagger = 1 - q^2$, one has

$$\mathcal{E}_J^\dagger(\mathsf{A}_F) = \mathcal{E}_{J,p^\dagger}(\mathsf{A}_F) = \Big(\prod_{e\in F} q_e\Big)\langle\sigma(F)\rangle_{J^*}, \tag{4.23}$$

where σ is an abstract Ising model on the dual HT structure $(E, \mathcal{F}_G^*)$, and the coupling constants J^* are the dual one to J, i.e. connected by the relationship $\tanh(J^*) = e^{-2J}$.

Proof. Upon conditioning first on $\xi \sim \mathcal{E}_J$, one sees that

$$\mathcal{E}_{J,p}(\mathsf{A}_F) = \sum_{\eta\subseteq E} \mathcal{E}_J(E \setminus \eta) r_{F\setminus\eta}.$$

Note that (4.22) follows because, by (4.17), $\mathcal{F}_G^*$ is the support of $\mathcal{E}_J$ and the weight of $\eta \in \mathcal{F}_G^*$ for $\mathcal{E}_J$ is proportional to q_η. When $r = q^2$, (4.23) follows from the high temperature expansion (4.20) and the fact that $q_\eta \times q_{F\setminus\eta}^2 = q_F \times q_{\eta\Delta F}$. □

Fix $e \in E$. We are now need interested in the function $q_e \mapsto \mathcal{E}_{J,p}(\mathsf{A}_F)$. In fact, we show it is a Moebius map, i.e., it is of the form $q_e \mapsto \frac{aq_e+b}{cq_e+d}$.

Corollary 1. *In the same context as Proposition 4.15, fix $e \in E$. The function $q_e \mapsto \mathcal{E}_{J,p}(\mathsf{A}_F)$ is a Moebius map. In particular, its derivative is of constant sign, and in the case that q and p are related by the relation $p = 1 - q^2$, its sign is equal to the sign of*

$$\left(\sum_{\eta\in\mathcal{F},e\notin\eta} q_\eta\right)\left(\sum_{\eta\in\mathcal{F},e\in\eta} q_{\eta\Delta F}\right) - \left(\sum_{\eta\in\mathcal{F},e\in\eta} q_\eta\right)\left(\sum_{\eta\in\mathcal{F},e\notin\eta} q_{\eta\Delta F}\right). \tag{4.24}$$

where $\mathcal{F}$ is a shorthand for the dual HT structure $\mathcal{F}_G^*$ on G.

Proof. From (4.22) it is clear that the function is a Moebius map. For such maps, the derivative is $\frac{ad-bc}{(cq_e+d)^2}$; in particular its sign is constant and is the sign of $q_e(ad-bc)$ when one writes $\mathcal{E}_{J,p}(\mathsf{A}_F) = \frac{aq_e+b}{cq_e+d}$, which is

$$\left(\sum_{\eta\in\mathcal{F},e\notin\eta} q_\eta\right)\cdot\left(\sum_{\eta\in\mathcal{F},e\in\eta} q_\eta r_{F\setminus\eta}\right) - \left(\sum_{\eta\in\mathcal{F},e\in\eta} q_\eta\right)\cdot\left(\sum_{\eta\in\mathcal{F},e\notin\eta} q_\eta r_{F\setminus\eta}\right).$$

In the case where $p = 1 - q^2$, i.e. $r = q^2$, we can use once again $q_\eta q^2_{\mathcal{F}\setminus\eta} = q_F q_{\eta\Delta F}$ to factorize the term above by $q_F^2 > 0$ so that the relevant sign is the sign of (4.24). □

4.4.3. *A weaker version of dual weak domination at* $p^\dagger$. In this section, we prove the following inequalities, that will be instrumental to prove the dual weak (hence weak†) domination and FKG for $p < p^\dagger$.

Proposition 4.16 (Inequalities at level $p^\dagger$)**.** *Take* $J \in (\mathbb{R}^+)^E$ *and* $p^\dagger = p^\dagger(J) = 1 - e^{-4J}$. *Then*

- (Almost dual weak FKG) *For any* $F, F' \subseteq E$
$$\mathcal{E}^\dagger_J(\mathsf{A}_F \cap \mathsf{A}_{F'}) \geq \mathcal{E}^\dagger_J(\mathsf{A}_F)\mathcal{E}^\dagger_J(\mathsf{A}_{F'}). \tag{4.25}$$
- (Almost dual weak domination) *For any* $J' \geq J$ *and* $F \subset E$, *we have*
$$\mathcal{E}_{J,p^\dagger}(\mathsf{A}_F) \geq \mathcal{E}_{J',p^\dagger}(\mathsf{A}_F). \tag{4.26}$$

The proofs of these two inequalities rely on the theory of abstract Ising model. We start by proving (4.25) which is shorter.

Proof of (4.25). We assume that F, F' are disjoint, as when this is not the case, replacing F' by $F'\setminus(F \cap F')$ yields a stronger inequality. Using (4.23) and the fact that $\mathsf{A}_F \cap \mathsf{A}_{F'} = \mathsf{A}_{F\Delta F'}$ we see that (4.25) is equivalent to $\langle\sigma(F\Delta F')\rangle_{J^*} \geq \langle\sigma(F)\rangle_{J^*}\langle\sigma(F')\rangle_{J^*}$, which follows from GKS inequality (Proposition 4.12). □

We now turn to the proof of (4.26). It relies also on Corollary 1.

Proof of (4.26). It is convenient to prove first the special case of the coupling constants J' is larger than J precisely on one edge, as states the following claim:

Claim 4.17. *For any fixed* $e \in E$, (4.26) *is true if* $J' = J$ *on every edge but* e.

Proof of the claim. Take $e \in E$ and recall the notation $q = e^{-2J}$, $q' = e^{-2J'}$ and note that by assumption $q_{\tilde{e}} = q'_{\tilde{e}}$ for all $\tilde{e} \in E\setminus\{e\}$, and $q_e > q'_e$. Let us define the function f on $[0, 1]$ as $f(x) = \mathcal{E}_{J(x),\tilde{p}}(\mathsf{A}_F)$ where the coupling constants $J(x)$ are defined as

$$J(x)_{e'} = \begin{cases} J_{e'} & \text{if } e' \neq e \\ -\log(x)/2 & \text{if } e' = e \end{cases} \qquad \text{so that } x = \exp(-2J(x)_e).$$

In term of this, what we want to show is precisely $f(q_e) \geq f(q'_e)$, which would follow from the fact that the function $x \mapsto f(x)$ is increasing. However, Corollary 1 ensures that the function f has derivative of constant sign given by (4.24), so what matters is to prove that

$$\left(\sum_{\eta\in\mathcal{F},e\notin\eta} q_\eta\right)\left(\sum_{\eta\in\mathcal{F},e\in\eta} q_{\eta\Delta F}\right) - \left(\sum_{\eta\in\mathcal{F},e\in\eta} q_\eta\right)\left(\sum_{\eta\in\mathcal{F},e\notin\eta} q_{\eta\Delta F}\right) \geq 0. \tag{4.27}$$

To prove this last inequality, define $\mathcal{F}^e = \{\eta \in \mathcal{F} : \ e \notin \eta\}$ and note that if $\mathcal{F}^e = \mathcal{F}$, it is trivial as two of the sums are 0. So assume $\mathcal{F}^e \neq \mathcal{F}$ and take $\eta_0 \in \mathcal{F} \setminus \mathcal{F}^e$. By GKS inequality (Proposition 4.12) for the HT structure $\mathcal{F}^e$ on E with coupling constants J^*, we have

$$\langle\sigma(\eta_0\Delta F)\rangle^{\mathcal{F}^e}_{J^*} \geq \langle\sigma(\eta_0)\rangle^{\mathcal{F}^e}_{J^*}\langle\sigma(F)\rangle^{\mathcal{F}^e}_{J^*}.$$

This inequality can be developed and expanded using the High Temperature expansion (4.20) to become

$$\left(\sum_{\eta\in\mathcal{F}^e} q_\eta\right)\left(\sum_{\eta\in\mathcal{F}^e} q_{\eta\Delta\eta_0\Delta F}\right) - \left(\sum_{\eta\in\mathcal{F}^e} q_{\eta\Delta\eta_0}\right)\left(\sum_{\eta\in\mathcal{F}^e} q_{\eta\Delta F}\right) \geq 0.$$

Since $\mathcal{F} = \mathcal{F}^e \bigsqcup (\mathcal{F}^e\Delta\eta_0)$ (indeed, either $\eta \in \mathcal{F}$ does contain e or does not, in which respective case it is in $\mathcal{F}^e\Delta\eta_0$ or in $\mathcal{F}^e$) so that the inequality above really is (4.27). This concludes the proof of the claim. □

Let us now explain how to deduce the general statement where we have two coupling constants $J \le J'$ that may differ on various edges from Claim 4.17: one can define a finite sequence $J(1), \dots, J(n)$ of coupling constants such that $J(1) = J, J(n) = J'$ and, for any $1 \le k \le n-1$, $J(k) \le J(k+1)$ but with equality on every edge but one. Writing $p(k) = 1 - \exp(-4J(k))$, the claim implies that for all $F \subset E$

$$\mathcal{E}_{J(k),p(k)}(\mathsf{A}_F) \ge \mathcal{E}_{J(k+1),p(k)}(\mathsf{A}_F).$$

However, since $p(k) \ge p(1) = \tilde{p}$ for any k (as $J \mapsto p^\dagger(J)$ is an increasing function), Proposition 3.5 implies for all subsets $F \subseteq E$

$$\mathcal{E}_{J(k),\tilde{p}}(\mathsf{A}_F) \ge \mathcal{E}_{J(k+1),\tilde{p}}(\mathsf{A}_F).$$

Whence (4.26) follows by chaining these inequalities. This concludes the proof of Proposition 4.16. □

In fact, at $p^\dagger$ there is also a slightly more general version of the FKG like inequality, as stated in the following proposition. Though this result is not needed elsewhere, we felt that it deserved to be written down.

Proposition 4.18. *For any $F, F' \subseteq E$*

$$\mathcal{E}^\dagger_J(\mathsf{A}_F \cap \forall_{F'}) \le \mathcal{E}^\dagger_J(\mathsf{A}_F)\mathcal{E}^\dagger_J(\forall_{F'}). \tag{4.28}$$

Proof. As explained in the proof of (4.25), it is enough to treat the case of F, F' disjoint, and the idea is to show a formula for $\mathcal{E}^\dagger_J(\mathsf{A}_F \cap \forall_{F'})$ that generalizes (4.23): it is possible to derive similarly as for (4.23) the equality

$$\mathcal{E}^\dagger_J(\mathsf{A}_F \cap \forall_{F'}) = q_F \frac{\sum_{\eta \in \mathcal{F}^{F'}} q_{\eta \Delta F}}{\sum_{\eta \in \mathcal{F}} q_\eta} \prod_{e \in F'} (1 - q_e^2) \tag{4.29}$$

where $\mathcal{F}^{F'} = \{\eta \in \mathcal{F} : \eta \cap F' = \emptyset\}$ is a subgroup of $\mathcal{F}$.

By Proposition 4.13, we have that

$$\langle \sigma(F) \rangle_{J^*}^{\mathcal{F}^{F'}} \le \langle \sigma(F) \rangle_{J^*}^{\mathcal{F}} \tag{4.30}$$

where on the left hand side the HT structure is $\mathcal{F}^{F'}$ and on the right it is $\mathcal{F}$; using High Temperature expansion (4.20), this can be rewritten

$$\frac{\sum_{\eta \in \mathcal{F}^{F'}} q_{\eta \Delta F}}{\sum_{\eta \in \mathcal{F}} q_\eta} \le \frac{\sum_{\eta \in \mathcal{F}} q_{\eta \Delta F}}{\sum_{\eta \in \mathcal{F}} q_\eta} \times \frac{\sum_{\eta \in \mathcal{F}^{F'}} q_\eta}{\sum_{\eta \in \mathcal{F}} q_\eta}. \tag{4.31}$$

However, upon multiplying by $q_F \prod_{e \in F'}(1 - q_e^2)$, we get (by the general equation (4.29)) on the left $\mathcal{E}^\dagger_J(\mathsf{A}_F \cap \forall_{F'})$ and on the right $\mathcal{E}^\dagger_J(\mathsf{A}_F \cap \forall_\emptyset)\mathcal{E}^\dagger_J(\mathsf{A}_\emptyset \cap \forall_{F'}) = \mathcal{E}^\dagger_J(\mathsf{A}_F)\mathcal{E}^\dagger_J(\forall_{F'})$, which concludes the proof of (4.28). □

4.4.4. *End of the proof of Theorem 4.5: weak† domination and weak† FKG below the threshold.* Now we come to the end of the proof of Theorem 4.5, that is weak† domination below the threshold $p^\dagger = p^\dagger(J)$. Of course, Proposition 4.16 has put us really close to it, and what remains to be done is to show that these inequalities allow us to prove weak† domination and weak† FKG strictly below the threshold $p^\dagger$.

Proposition 4.19. *Under the notations of Theorem 4.5, if $p \in [0,1]^E$ is a parameter such that $p < p^\dagger(J)$, then for every $J' \ge J$,*

$$\mathcal{E}_{J,p} \preccurlyeq^c \mathcal{E}_{J',p}. \tag{4.32}$$

Furthermore, $\mathcal{E}_{J,p}$ has dual weak FKG.

Before proving this proposition, which concludes the proof of Theorem 4.5, let us remark that the case of a non-connected graph reduces to the case of a connected one. The reason for this is that when G is a non-connected graph, all the measures that we are dealing with factorize as independent measures restricted to each connected components of the graph, and if the properties we want to show hold for each of these components, then they hold as well for G itself.

The same is true for "loop-edges", i.e. those edges whose endpoints are the same vertex: the percolation on them is independent of the rest as well. Therefore, without loss of generality, during the proof we assume that G is connected without loop edges.

We make one more reduction, which removes the possible tree part of the graph. Declare two edges $e, e' \in E$ to be equivalent if either $e = e'$ or there exists a cycle of G containing both of them; this is the connected-component relation of the cycle matroid of G. Bridges form singleton classes. In the variables $\tau_{xy} = \sigma_x\sigma_y$ (or, equivalently, $\xi_{xy} = (1 + \tau_{xy})/2$), the only constraints are the cycle constraints $\prod_{e\in C} \tau_e = 1$. Since each cycle is contained in one of the above classes and the Ising weight factorizes over edges, the law of the energy field factorizes as a product over these classes. This factorization is preserved after intersection with Bernoulli percolation, after taking complements, and when the couplings are changed. Therefore it is enough to prove the proposition on each class separately: stochastic domination tensorizes, and the product of FKG measures is FKG. The one-edge bridge classes are immediate Bernoulli cases. We may thus assume in the proof below that, apart from the trivial one-edge case, every two distinct edges of G lie on a common cycle.

Proof. Thanks to Proposition 3.5 (resp. Proposition 3.9), to prove dual weak domination (4.32) (resp. dual weak FKG for $\mathcal{E}_{J,p}$) it is enough to show that (4.26) (resp. (4.25)) is strict for $F = \{e\}$ a singleton (resp. for $F = \{e\}, F' = \{e'\}$ both singletons).

We now use that, when F is a singleton, $\mathsf{A}_F = (\forall_F)^c$ and thus, $\mathcal{E}_{J,p}(\mathsf{A}_F) = 1 - p\mathcal{E}_J(\forall_F)$. It is therefore equivalent to prove that $\mathcal{E}_J(\forall_F) < \mathcal{E}_{J'}(\forall_F)$ (resp. $\mathcal{E}_J(\forall_{F\cup F'}) > \mathcal{E}_J(\forall_F)\mathcal{E}_J(\forall_{F'})$), that is to prove two statements purely on Ising's energy field.

However, these strict inequalities can be reinterpreted in term of classical Ising correlations. For the first one, i.e. $\mathcal{E}_J(\forall_F) < \mathcal{E}_{J'}(\forall_F)$, if $F = \{e\} = \{(xy)\}$ then $\mathcal{E}_J(\forall_F) = \mathcal{E}_J[\xi_e] = \frac{1}{2}(\langle\sigma_x\sigma_y\rangle_J + 1)$ and the inequality rewrites $\langle\sigma_x\sigma_y\rangle_J < \langle\sigma_x\sigma_y\rangle_{J'}$, i.e. the fact that the two point function is *strictly* increasing in any of the coordinate of the coupling constant J.

For the second one, i.e. $\mathcal{E}_J(\forall_{F\cup F'}) > \mathcal{E}_J(\forall_F)\mathcal{E}_J(\forall_{F'})$, if $F = \{(xy)\}$ and $F' = \{(x'y')\}$, it can be rewritten in the form

$$\langle\sigma_x\sigma_y\sigma_{x'}\sigma_{y'}\rangle_J - \langle\sigma_x\sigma_y\rangle_J \cdot \langle\sigma_{x'}\sigma_{y'}\rangle_J = 4\mathrm{Cov}_J\big(\xi(\sigma)_e, \xi(\sigma)_{e'}\big) > 0, \tag{4.33}$$

i.e. that GKS inequality is strict.

Now, it can be remarked that the first inequality is actually a corollary of the second one since when one derives $\langle\sigma_x\sigma_y\rangle_J$ as a function of $J_{e'}$, one obtains the covariance above (see e.g. [FV17]). To get the strict positivity of (4.33), a close check to the proof of GKS inequalities shows that after the reduction performed above, the GKS inequality is indeed strict between four points x, y, x', y' corresponding to two edges. Indeed, the two edges are part of a cycle, and it can be seen from the standard high-temperature proof that one obtains a strictly positive contribution by using the two arcs of that cycle. □

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